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Conservation of acceleration and dynamic entanglement in mechanics

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Abstract Within a flow of a viscous fluid the effects of compression and viscosity are closely coupled with inertia. Discrete mechanics gives a physical sense of these interactions by showing that they exist only in a dynamic vision where the variation of the velocity in time generates the entanglement of the effects of compression and shearing. These two phenomena are described in the form of a Helmholtz–Hodge decomposition by two orthogonal terms within the discrete equation of motion, the first curl-free and the second divergence-free. They can only exchange mechanical energy if the acceleration is nonzero. This entanglement, which occurs at all spatial scales, is a function of longitudinal and transverse celerities. After a presentation of the formal framework, simple examples allow to understand the temporal shifts of direct and induced flows in accordance with the causality principle.

Mathematics Subject Classification 76A02 · 76F06

1 Introduction

Dynamic entanglement can be defined as an indirect interaction due to a variation in time of a common variable. The oldest example of such a phenomenon is probably that of Maxwell [17], who explained that the direct and induced currents were coupled by the temporal variation of one of the electromagnetic fields. He unified the laws of magnetism and electricity, including the present form of four equations that bear his name.

In the field of mechanics, the equations of fluid mechanics and those of solid mechanics are still not unified, even though monolithic models exist in fluid–structure interaction. Several differences remain between the equations of these two domains. This is the case for the very important inertial terms in fluid, in particular for the study of turbulence integrated into the Navier–Stokes equation and often neglected in the Navier–Lamé equation. The second important difference concerns the addition of the conservation of mass to the Navier–Stokes equation, which does not inherently conserve either mass or energy. Indeed, the Navier–Stokes equation

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is always linked to an adjoint equation, the conservation of mass for both incompressible and compressible formulations. The Navier–Stokes equation cannot be solved without the use of the adjoint equation. This decoupling between the equation of motion and the mass conservation constraint can have repercussions when the phenomena are dynamic, at high Reynolds number for example.

On the other hand, the Navier–Lamé equation is not associated with another conservation law, but the Lamé coefficients, λ and μ outside the operators, inhibit the possibility of really splitting the compressive effects of shear.

Discrete mechanics introduced an alternative derivation of the equation of motion on the basis of existing principles, some of which have been reinterpreted by introducing the geometric nature absent from the concept of continuous medium. It establishes two primal and dual topologies on which the modeling of the effects of compression and shearing is based, which gives them essential properties, in particular local and global orthogonality [5,8]. The fundamental principles of physics are revisited: principle of equivalence, principle of relativity, equivalence of mass and energy, etc. The resulting law of motion excludes the notion of mass or density and becomes purely kinematic, with the constitutive relations appearing only through the potentials of acceleration.

The discrete formulation is very coherent with the bases of differential geometry and numerical methodologies to solve the problems of mechanics in terms of velocity and potentials of acceleration. The equation of motion, which has been recently used to deal with fluid mechanics problems, fluid–structure interactions, heat transfers or wave propagation, is used in this context to analyze in depth the dynamic entanglement between the effects of compression and shear in mechanics.

The recently developed discrete mechanics [6,8] shows that it is possible to reformulate the equations of fluid and solid mechanics in a fluid–structure approach that can be considered as an alternative to the classical equations. Although this formulation based on differential geometry leads to a significantly different law, the results obtained on many analytical solutions and simulations are strictly the same. Two-phase flows simulated with the discrete approach in the presence of jump conditions related to capillary effects also recovered a large number of known solutions [5]. One of the fundamental differences in the discrete mechanics law concerns the inertia term where the Lamb vector of the Navier–Stokes equation is replaced by a rotational [4]. This Helmholtz–Hodge decomposition of the inertia terms gives symmetry and invariance properties to this description.

After a brief presentation of the discrete formulation, the interlocking process of the direct and induced effects of the equation of motion is described and illustrated with classical examples from mechanics. This classification into two terms of gradient of a scalar potential and rotation of a vector potential, respectively, is reflected in all other terms of the equation, especially for the nonlinear inertia term. These specific reasons for the presented formulation can lead to different behaviors when the mechanical stress state is large and the system evolves in time. This is the case in fluid mechanics for a turbulent Taylor–Green flow where the evolution of the kinetic energy over time is not the same as that obtained from the Navier–Stokes equations even if quantitatively the energy cascade is the same.

2 Discrete mechanics framework

2.1 Physical principles revisited

The principles of physics and mechanics are not called into question; they will simply be revisited from the discrete angle chosen to derive an equation of motion which is presented as an alternative to the equations of the mechanics of continuous media.

One of the oldest principles, dating from the beginning of the sixteenth century and initiated in particular by Galileo, expresses the equivalence between inertial mass and gravitational mass; the acceleration of a body is then independent of its nature. This principle, since titled the weak equivalence principle, was translated by Newton in the form of his second law, the fundamental law of dynamics, $m\,\boldsymbol{\gamma} = \mathbf{F}$ [19]. This logical interpretation in the historical context, where gravitation was the dominant theory of the preoccupations of the time, still exists today in many theories such as that of relativity, fluid flows. Mass is attached to other quantities like energy ($e = m\,c^2$) or momentum $\mathbf{q} = m\,\boldsymbol{v}$. The Navier–Stokes equation is presented as a conservation of momentum. The interpretation given in discrete mechanics is different: if the mass does not intervene on the acceleration of a body subjected to an external acceleration, then why should keep it? The postulate of discrete mechanics is to extend the equality of intrinsic and gravitational accelerations to any other acceleration. The

law of discrete dynamics expresses the fact that the acceleration of a material medium or of a particle with or without mass is equal to the sum of the external accelerations; it is written:

$$\boldsymbol{\gamma} = \boldsymbol{h} \tag{1}$$

where $\boldsymbol{h}$ is the sum of all accelerations applied to the particle or to the material medium. This law is intrinsically valid in accelerated rectilinear translation motion, independently of the velocity. Law (1) expresses the conservation of acceleration on a rectilinear support. Acceleration is a quantity that can be measured anywhere, it is an absolute quantity, while velocity, force, energy, etc., are relative quantities, the knowledge of which in a previous state is required.

The Galileo principle of relativity, where all the laws of mechanics are identical in all the Galilean frames of reference, is interpreted in a different way since the notion of global frame of reference has been abandoned. The current velocity is here set from the expression $\boldsymbol{v} = \boldsymbol{v}^o + \boldsymbol{\gamma}\, \mathrm{d}t$, where $\boldsymbol{v}^o$ is the velocity at the initial time and $\mathrm{d}t$ the considered time increment. If the acceleration is zero, the velocity is kept on a rectilinear path. The absolute velocity $\boldsymbol{v}^o$ of a body at uniform velocity is not required, it must naturally disappear from the equation of motion. The uniform rotational motion, which is a little different, also satisfies this principle of relativity under certain conditions [4]. The discrete law of motion must then integrate the temporal process necessary to deduce the state of a system at time t from that at time t^o; the proposed model is associated with continuous memory.

The notion of tensor initiated by mechanicians for mechanics is also abandoned. It was linked to the nature of materials whose properties, depending on the directions of space, can be expressed simply by tensors. In discrete mechanics the notion of global frame of reference is replaced by a local frame of reference on a discrete geometric topology.

The principle of causality must of course be satisfied in the development of any physical model. In discrete mechanics this principle is satisfied if we consider that a piece of information can only be perceived over a distance $\mathrm{d}h$, the discrete horizon, if $\mathrm{d}t$ is greater than $\mathrm{d}t = \mathrm{d}h/c$ where c is the celerity of the longitudinal wave. This parameter is intrinsic to the considered medium, for example the celerity c_0 of light in vacuum or the celerity of sound in material media, fluid or solid. In a system with several dimensions it is important to know how information is transmitted in a material medium, as this conditions the entanglement of the different mechanical effects.

Finally, the Helmholtz–Hodge decomposition, which is not really a principle, consists in separating the curl-free component and the divergence-free component of any vector. In mathematics, the Helmholtz–Hodge decomposition is useful for projecting a vector onto a space with zero divergence [1,23,26]. In fluid mechanics, it easily ensures the incompressibility constraint [13]. Numerous publications on the Helmholtz–Hodge decomposition [1] show its theoretical interest and the potentialities of its applications. This decomposition of acceleration a priori is one of the cornerstones of discrete mechanics, and it makes it possible to separate the terms of compression from the terms of rotation describing the motion of a material medium.

The physical parameters are essentially the longitudinal and transverse celerities, which must be located precisely within the discrete frame of reference. Indeed, the framework of classical mechanics is not adapted to a discrete vision based on these physical principles, in particular because of the concept of continuous medium where all the quantities are defined in a point. For this reason it is essential to define a discrete geometric topology from which the basic phenomena are described.

2.2 Primal and dual geometric topologies

The derivation of the discrete equation of motion requires a complete geometric framework; the notions of continuous medium, derivation at a point, analysis and continuum mechanics are abandoned and replaced by two topologies called primal and dual, which can be inverted. The privileged directions of classical mechanics, like the notion of global frame of reference, are also abandoned. Figure 1 shows schematically the two elementary geometric topologies; their orientation in space is not specified—one will simply define a local frame of reference by the orientations of the unit vectors $\mathbf{t}$ of the segment Γ and $\mathbf{n}$ orthogonal to the primal facet $\mathcal{S}$.

The oriented contour Γ^* of the primal facet is formed by the set of sides of the corresponding polygon; the primal $\mathcal{S}$ and dual Δ facets are considered to be planar, even if this is not always the case in practice. The contour of the dual facet is made up of all the oriented segments δ of the associated polygon. The vectors $\mathbf{t}$ and $\mathbf{n}$ are orthogonal by construction, $\mathbf{t} \cdot \mathbf{n} = 0$. The segment Γ of length $\mathrm{d}h = [a, b]$, whose ends are a and b, is chosen to represent the component of velocity $\boldsymbol{v}$ and the component of acceleration $\boldsymbol{\gamma}$. The velocity vector

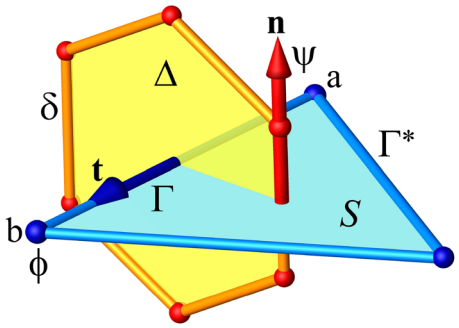

Fig. 1 A primitive planar facet $\mathcal{S}$ defined by a contour Γ^* is oriented according to the unit normal $\mathbf{n}$ such that $\mathbf{n} \cdot \mathbf{t} = 0$; the planar dual surface Δ connecting the centroids of the cells is defined by the dual contour δ

and the acceleration vector will remain undefined; even if they can be reconstructed using their components, this will not be useful. Two functions will be used subsequently, ϕ a scalar defined at points a and b named scalar potential and $\boldsymbol{\psi}$ associated with the unit vector $\mathbf{n}$ named vector potential. It is necessary to specify the notion of discrete velocity $\boldsymbol{v}$; this constant quantity on the segment Γ represents the average velocity of the local velocity $\boldsymbol{w}$ on it:

$$\boldsymbol{v} = \frac{1}{\mathrm{d}h} \int_a^b \boldsymbol{w} \cdot \mathbf{t} \, \mathrm{d}l \tag{2}$$

The local velocity has therefore a linear variation on the segment and takes of course the values $\boldsymbol{w}_a$ and $\boldsymbol{w}_b$ at its extremities. The real unknowns remain the n_e average velocities on the n_e segments of the primitive structure. The kinetic energies on these vertices $\boldsymbol{w}_a^2/2$ and $\boldsymbol{w}_b^2/2$ are reconstructed from the velocities $\boldsymbol{v}$ associated with a vertex. The divergence theorem transforms the sum of the fluxes, the product of each velocity component associated with the corresponding facet of the dual volume, into a divergence assigned to the vertex.

In addition, four discrete differential operators are introduced. The first is the operator $\nabla\phi$ applied to the scalar potential; it differs from the classical gradient vector because it only applies to the component on the segment Γ, i.e., $\nabla\phi = (\phi_b - \phi_a)/d$. Thus $\nabla\phi$ is both a scalar on the oriented segment and a vector; it is the restriction of the gradient of a classical scalar function to the segment Γ, $\nabla\phi = \nabla\phi \cdot \mathbf{t}$. The second operator, named primal curl, is calculated from the circulation of the vector $\boldsymbol{v}$ on the contour Γ^* using the Stokes' theorem, $\nabla \times \boldsymbol{v}$; it will then be assigned to a dual segment δ and named $\boldsymbol{\psi}$, one of the curl components orthogonal to $\mathbf{n}$. Similarly, the dual-curl operator $\nabla^d \times \boldsymbol{\psi}$ corresponds to a line integral on the dual contour and then assigned to the primal segment Γ. The dual volume is composed by the collection of dual facets Δ associated with the considered vertex, a or b in Fig. 1. Finally, the divergence operator corresponds to the integral on the dual volume of all the fluxes having the same vertex, $\nabla \cdot \boldsymbol{v}$; the result is then assigned to the vertex a. In a continuous context, $\boldsymbol{v}$ is a polar vector and $\boldsymbol{\psi}$ an axial or pseudo-vector. The two primal and dual-curl operators are thus differentiated by the function to which they apply. This discrete framework is completed by two notions, (i) the time lapse $\mathrm{d}t = t - t^o$ between the current time and the time t^o where the initial equilibrium is defined, and (ii) the discrete horizon $\mathrm{d}h = c \, \mathrm{d}t$, the length traveled by a wavefront of sound celerity c on a rectilinear trajectory.

The physical model has many properties, in particular the local and global orthogonality of the operators, $\nabla\phi \cdot \nabla^d \times \boldsymbol{\psi} = 0$, also verified by the discrete operators; the discrete model also mimics the properties of the continuum $\nabla \cdot \nabla^d \times \boldsymbol{\psi} = 0$ and $\nabla \times \nabla\phi = 0$, whatever the polygonal or polyhedral geometric topologies, structured or not structured [8].

2.3 Conservation of acceleration

The conservation of acceleration on a segment Γ, considered as a postulate of discrete mechanics, can be strongly related to the conservation of energy. Indeed, acceleration $\boldsymbol{\gamma}$ on segment Γ in Fig. 1 is constant and its integration between a and b is none other than the variation of the total energy per unit of mass Φ between these two points:

$$\Phi_b - \Phi_a = \int_a^b \boldsymbol{\gamma} \cdot \mathbf{t} \, \mathrm{d}l \tag{3}$$

Even if the velocity is not zero, the fact that it is also constant on the segment leads to zero energy variation according to the principle of relativity; all the terms of the particle derivative, the time derivative and the inertia are indeed equal to zero. The equivalence between mass and energy of the theory of relativity must be verified in this discrete context. It is therefore possible to impose the conservation of energy without having to additionally require the conservation of mass. The choice fixed in this context is to conserve the total energy and to eliminate the notion of mass, in accordance with the principle of discrete equivalence described above.

The acceleration vector, or its component $\boldsymbol{\gamma}$, is decomposed into a curl-free part and another divergence-free part according to the Helmholtz–Hodge decomposition:

$$\boldsymbol{\gamma} = -\nabla\phi + \nabla^d \times \boldsymbol{\psi} \tag{4}$$

where ϕ is the scalar potential of the acceleration located at the points of the primal topology and $\boldsymbol{\psi}$ its vector potential assigned on the segment δ of normal unit $\mathbf{n}$.

These two components are assigned to the same segment Γ as the acceleration itself, which is their sum. It is essential to note that these two components have no direct action on each other, they simply circulate on the same trajectory. The Helmholtz–Hodge decomposition is sometimes written with a third harmonic component which is both curl-free and divergence-free; in the present case, this last term is deleted because it corresponds to the uniform movements of translation and of rotation separated from the outset by the discrete differential operators. The energy conservation between a and b is then written in the form of a discrete integral:

$$\int_\Gamma \boldsymbol{\gamma} \cdot \mathbf{t}\, \mathrm{d}l = -\int_\Gamma \nabla\phi \cdot \mathbf{t}\, \mathrm{d}l + \int_\Gamma \nabla^d \times \boldsymbol{\psi} \cdot \mathbf{t}\, \mathrm{d}l \tag{5}$$

where ϕ is the scalar potential of the acceleration located at the points of the primal topology and $\boldsymbol{\psi}$ its vector potential assigned on the segment δ of normal unit $\mathbf{n}$.

Local (4) and integral (5) laws thus represent the conservation of energy and consequently the conservation of mass. The direct action represented by $-\nabla\phi$ cannot be modified by the induced action defined by $\nabla^d \times \boldsymbol{\psi}$, or vice versa, because these two components are orthogonal. The entanglement resides in the fact that the intrinsic acceleration $\boldsymbol{\gamma}$ is necessary to allow the exchanges between the two orthogonal components. This entanglement does not depend on the spatial scale and can act at a great distance, but the effects depend on celerities c_l and c_t.

Several equations in physics present this dynamic entanglement, in solid or fluid mechanics or in electromagnetism. Some do not, for example Darcy's equation which derives only from a scalar potential ϕ_D, such that $\boldsymbol{v} = -\nabla\phi_D$. The generalized Darcy equation $\partial\boldsymbol{v}/\partial t = -\nabla p - \mu/K\, \boldsymbol{v}$ becomes $\partial\boldsymbol{v}/\partial t = -\nabla\,(p + \phi_D)$. The viscous effects act at a very short spatial scale of order of magnitude of $\sqrt{K}$, being the permeability of the porous medium.

2.4 Discrete motion equation

The derivation of the discrete law of motion [3] serves to establish an equation to predict the solution of a problem at current time t from that at time t^o, which represents the state of the physical system in mechanical equilibrium at an earlier instant. The discrete law defines a continuous memory model where the current mechanical equilibrium is calculated by a process of accumulation of compressive and shear stresses. The potentials at instant t^o, ϕ^o and $\boldsymbol{\psi}^o$, are energies per unit mass accumulated between the initial instant and instant t^o; they are called retarded potentials with reference to those of Liénard and Wichert [15]. The local equation of motion is then written as:

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\nabla\left(\phi^o + d\phi\right) + \nabla^d \times \left(\boldsymbol{\psi}^o + d\boldsymbol{\psi}\right) \tag{6}$$

The increments of potentials $\mathrm{d}\phi$ and $\mathrm{d}\boldsymbol{\psi}$ must be modeled with fundamental compression and shear experiments in mind. The variation of the scalar potential is related to the compression characterized by the divergence of the velocity and to the longitudinal celerity in the form $\mathrm{d}\phi = -\mathrm{d}t\; c_l^2\; \nabla \cdot \boldsymbol{v}$. The variation of the vector potential depends on the curl of the velocity and the transverse celerity $\mathrm{d}\boldsymbol{\psi} = -\mathrm{d}t\; c_t^2\; \nabla \times \boldsymbol{v}$. The balance of acceleration (5) on the segment Γ becomes:

$$\int_\Gamma \boldsymbol{\gamma} \cdot \mathbf{t}\, \mathrm{d}l = -\int_\Gamma \nabla\left(\phi^o - \mathrm{d}t\; c_l^2\; \nabla \cdot \boldsymbol{v}\right) \cdot \mathbf{t}\, \mathrm{d}l + \int_\Gamma \nabla^d \times \left(\boldsymbol{\psi}^o - \mathrm{d}t\; c_t^2\; \nabla \times \boldsymbol{v}\right) \cdot \mathbf{t}\, \mathrm{d}l \tag{7}$$

The retarded potentials reflect the storage of energy from the initial moment, or at least from a previous state of mechanical equilibrium, at moment t^o. The history of compressive and shear energies is represented by these two retarded potentials:

$$\phi^o = -\int_0^{t^o} c_l^2 \, \nabla \cdot \boldsymbol{v} \, \mathrm{d}\tau; \qquad \boldsymbol{\psi}^o = -\int_0^{t^o} c_t^2 \, \nabla \times \boldsymbol{v} \, \mathrm{d}\tau \tag{8}$$

The equation of local motion can then be extracted by considering that distance $dh = [a, b]$ tends to zero. However, unlike the notion of continuous medium where all the quantities are reduced to a point, discrete mechanics interprets this step as a homothetic decrease toward zero of the distances, which preserves the angles between the segments of the geometric topologies. The law of motion and the temporal updates of potential read:

$$\begin{cases} \boldsymbol{\gamma} = -\nabla \left(\phi^o - c_l^2 \, \mathrm{d}t \, \nabla \cdot \boldsymbol{v} \right) + \nabla^d \times \left(\boldsymbol{\psi}^o - c_t^2 \, \mathrm{d}t \, \nabla \times \boldsymbol{v} \right) + \boldsymbol{h}_s \\ \alpha_l \, \phi^o - c_l^2 \, \mathrm{d}t \, \nabla \cdot \boldsymbol{v} \longmapsto \phi^o \\ \alpha_t \, \boldsymbol{\psi}^o - c_t^2 \, \mathrm{d}t \, \nabla \times \boldsymbol{v} \longmapsto \boldsymbol{\psi}^o \end{cases} \tag{9}$$

where $\boldsymbol{h}_s$ represents the source terms, gravitational acceleration, capillary acceleration, etc. The intrinsic acceleration of the material medium or of the particle will be expressed from the partial derivative in time and the terms of inertia. The quantities α_l and α_t between 0 and 1 are attenuation factors for longitudinal and transverse waves; when $\alpha = 0$, the waves persist indefinitely. The symbol $\longmapsto$ corresponds to the temporal update of the quantity associated between t^o and t; it is a discrete integration which traces the accumulation of mechanical stresses over time. The celerities can be measured directly on the medium by considering the propagation of waves or derived from relations with other quantities, Young's modulus, Poisson's ratio, etc. They must simply be known.

The material derivative $\mathrm{d}\boldsymbol{v}/\mathrm{d}t$ is modeled in a different way from those classically admitted in continuum mechanics; it was the subject of a specific presentation in [4]:

$$\boldsymbol{\gamma} = \frac{\partial \boldsymbol{v}}{\partial t} + \nabla \left(\frac{1}{2} \, |\boldsymbol{v}|^2 \right) - \nabla^d \times \left(\frac{1}{2} \, |\boldsymbol{v}|^2 \, \mathbf{n} \right) \tag{10}$$

The inertial terms are described by a Helmholtz–Hodge decomposition of the inertial potential $\phi_i = |\boldsymbol{v}|^2/2$. The sum of these potentials with those of the acceleration will be called Bernoulli potentials, $\phi_B^o = \phi^o + \phi_i$ and $\boldsymbol{\psi}_B^o = \boldsymbol{\psi}^o + \boldsymbol{\psi}_i$ where $\boldsymbol{\psi}_i = \phi_i \, \mathbf{n}$.

Unlike longitudinal waves, transverse waves are polarizable in any planar surface orthogonal to the primal facet around the unit normal $\mathbf{n}$. The shear in a given direction in the surface of the facet $\mathcal{S}$ induces a transverse wave of celerity c_t in the orthogonal direction $\mathbf{n}$. For example, for the plane startup Couette flow generated by a wall, the propagation of the transverse waves takes place in the direction orthogonal to the wall. In fluid, the depth of penetration of the waves is limited by viscous dissipation. It should be noted that for the Couette flow, the particular form of inertia in discrete mechanics leads, as in continuum mechanics, to zero inertia but only by compensation; the last two terms of relation (10) are indeed different from zero. As the Reynolds number increases, this flow becomes turbulent. In the general case of turbulent flows, the time constants of the vortex structures decrease and the interactions between the effects of pressure and shear are greater; these effects are closely intertwined with inertia.

The particular symmetrical form of equation of motion (9) in two Lagrangians induces according to Noether's theorem [14], the invariance of certain quantities, in particular the mechanical energy linked to temporal independence of physical law.

3 Principle of dynamic entanglement

The particular form of inertia separated in two terms of Helmholtz–Hodge decomposition (10) allows to gather all the curl-free terms and the divergence-free terms to the second member of Eq. (9). Thus the acceleration $\boldsymbol{\gamma} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t$ becomes a simple time derivative of the velocity which also takes the form of a Helmholtz–Hodge decomposition. The motion is stationary only if the two operators are equal $-\nabla\phi = \nabla^d \times \boldsymbol{\psi}$, and as they

are orthogonal, this equality leads to the existence of a harmonic function, both curl-free and divergence-free; the equation decouples into two distinct components. A contrario and, in the general case of unsteady motion, the two functions are entangled. This entanglement physically represents the exchange of compressional and rotational energies which can only be effective if the phenomenon is unsteady. The system of equations is then put in the form:

$$\begin{cases} \dfrac{\partial \boldsymbol{v}}{\partial t} = -\nabla\left(\phi^o + \dfrac{|\boldsymbol{v}|^2}{2} - c_l^2\,\mathrm{d}t\,\nabla\cdot\boldsymbol{v}\right) + \nabla^d\times\left(\boldsymbol{\psi}^o + \dfrac{|\boldsymbol{v}|^2}{2}\,\mathbf{n} - c_t^2\,\mathrm{d}t\,\nabla\times\boldsymbol{v}\right) + \boldsymbol{h}_s \\ \alpha_l\,\phi^o - c_l^2\,\mathrm{d}t\,\nabla\cdot\boldsymbol{v} \longmapsto \phi^o \\ \alpha_t\,\boldsymbol{\psi}^o - c_t^2\,\mathrm{d}t\,\nabla\times\boldsymbol{v} \longmapsto \boldsymbol{\psi}^o \end{cases} \tag{11}$$

The source term $\boldsymbol{h}_s = -\nabla\boldsymbol{\phi}_g + \nabla^d\times\boldsymbol{\psi}_g$ is written in the same form of two orthogonal contributions; whatever the source terms considered, gravitation, capillary effects, electromagnetic effects,…, this form is always possible. In discrete mechanics gravitation (or other effects) derives from two potentials, one scalar and the other vector.

First let us look at the case of nonentangled effects. If $\boldsymbol{\gamma} = 0$, we have $-\nabla\phi + \nabla^d\times\boldsymbol{\psi} = 0$ and as these two quantities are orthogonal they can only be equal to a harmonic function $\boldsymbol{h}$, both curl-free and divergence-free, $-\nabla\phi = \nabla^d\times\boldsymbol{\psi} = \boldsymbol{h}$. By applying, respectively, the differential operators divergence and curl, it becomes:

$$\begin{cases} \nabla^2\phi = 0 \\ \nabla^2\boldsymbol{\psi} = 0 \end{cases} \tag{12}$$

To recall a simple example, a differential equation of two functions of different variables allows a separation of variables and the two parts can only be equal to one constant. Both compression and shear effects are nonentangled; the potentials ϕ and $\boldsymbol{\psi}$ can be obtained separately from the boundary conditions. We understand why the Helmholtz–Hodge decomposition of any vector into two components, one with curl-free and the other with divergence-free, is impossible a priori in the general case because the functions $\nabla\cdot\boldsymbol{h}$ and $\nabla\times\boldsymbol{h}$ are only defined up to a curl (resp. gradient); knowledge only of the vector to be decomposed is not enough to guarantee the uniqueness of the solution.

The effects of compression and of shearing are tightly coupled via the derivative in time, and by associating the potentials of acceleration with those of inertia one has:

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\nabla\phi_B + \nabla^d\times\boldsymbol{\psi}_B \tag{13}$$

where $\phi_B = \phi^o + |\boldsymbol{v}|^2/2$ is the Bernoulli potential located on the vertices and $\boldsymbol{\psi}_B = \boldsymbol{\psi}^o + |\boldsymbol{v}|^2/2\,\mathbf{n}$ is the Bernoulli potential assigned to the primal facets. Thus it is the derivative in time which couples the two components of Bernoulli.

A new case assumes that the derivative in time is zero, but the inertial terms are different from zero. An example is steady incompressible flow with a Reynolds number greater than unity. As the problems are nonlinear, their solutions can only be obtained through numerical techniques like Newton's associated with a method ensuring the incompressibility constraint $\nabla\cdot\boldsymbol{v} = 0$. It is, however, easier to obtain a stationary solution under constraint by solving an unsteady problem leading to the solution independent of time. When the motion is steady, $\partial\boldsymbol{v}/\partial t = 0$, we return to the case of a nonentangled system where the gradient terms of the equation of motion are separately zero from those in dual curl. The solution then depends only on the variables $(\boldsymbol{v}, \phi_B, \boldsymbol{\psi}_B)$.

In the general case, this is Eq. (9) that must be integrated using the incremental process described; it is not possible to obtain a solution a priori without knowing the state of the physical system with reference to a previous state if the effects of compression and shear coexist. For example, it is possible to obtain the solution of the viscous Poiseuille flow in a channel if we assume incompressibility, as it is also possible to calculate the flow of an ideal gas in a nozzle in stationary supersonic regime without reference to a previous state. If the movement is divergence-free or curl-free straight away, then the effects are not entangled. But in the general case where the effects of compression and shear are present simultaneously, a stationary solution can only be obtained with a converging temporal process. For nonlinear and unsteady problems like those in fluid–structure interaction or for turbulent flows, the interweaving of the effects is complex and cannot be analyzed simply.

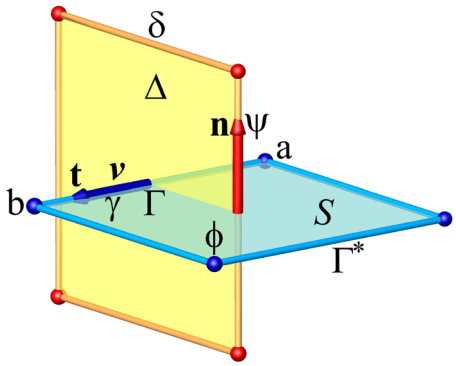

Fig. 2 The collection Γ^* composed of the segments Γ of unit vectors $\mathbf{t}$ forms the primal geometric topology $\mathcal{S}$. This facet is oriented according to the normal unit $\mathbf{n}$ such that $\mathbf{n} \cdot \mathbf{t} = 0$; the dual surface Δ connecting the centroids of the cells is defined by the dual contour δ

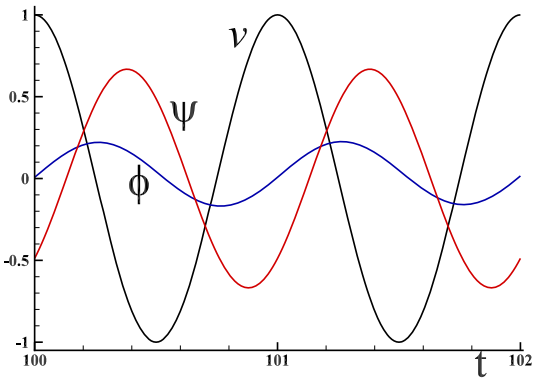

Fig. 3 Evolution in time of the component $\boldsymbol{v}$ imposed on the segment Γ and those of the scalar potential ϕ and vector $\boldsymbol{\psi}$

3.1 A simple example

Discrete equation (9) is used to obtain solutions of various problems in compressible or incompressible fluid mechanics, in two-phase flows, in solid mechanics or in fluid–structure interaction, in heat transfer, for wave propagation, etc. The resolution of these problems can be carried out on various geometric topologies based on polygons and polyhedra with any number of faces. For these real cases the number of cells can be high. The numerous examples treated with the help of this equation show a spatial and temporal convergence rate of order two [8].

In order to better exhibit the behavior of the physical model, the case of a very simple geometry is approached by considering a single primal face $\mathcal{S}$ associated with its dual face Δ, schematized in Fig. 2.

The oriented segment Γ defined by the unit vector $\mathbf{t}$ and its two extremities a and b with $\mathrm{d}h = [a, b] = 1$ carries the components $\boldsymbol{v}$ of the velocity and those of acceleration $\boldsymbol{\gamma}$. The scalar potential ϕ is located on the vertices of the segments, and the vector potential is carried by a segment of the dual topology of unit vector $\mathbf{n}$.

The velocity is imposed on one of the segments in the form $\boldsymbol{v} \cdot \mathbf{t} = V_0 \cos(\omega\, t)$ over a sufficiently long time so as to reduce the effects of the initial condition and get a purely periodic regime. The resolution of this problem is carried out with the system of Eq. (9) adapted to a fluid of longitudinal celerity c_l and a kinematic viscosity $\mathrm{d}t\, c_t^2 = \nu$ with $c_l = 1\ \mathrm{m\ s^{-1}}$ and $\nu = 10^{-2}\ \mathrm{m^2\ s^{-1}}$.

When a constant flow is maintained on the segment imposed by $V_0 = 1$ and $\omega = 0$, there is no interaction between the two topologies; the velocity in the cell is constant and equal to $\boldsymbol{v} = 1$ and the zero vector potential, $\boldsymbol{\psi} = 0$. On the other hand, when the imposed velocity is variable in time and $\omega = 2\,\pi$, the flows corresponding to accelerations $-\nabla\phi_B^o$ and $\nabla^d \times \boldsymbol{\psi}_B^o$ are also variable in time. From a zero solution and after a few periods, the solution is entirely periodic, with a period equal to $T = 1$. Figure 3 shows the evolution of $\boldsymbol{v}$ imposed and those of the potential ϕ^o and of $\boldsymbol{\psi}^o$.

We notice that the periodic evolution of ϕ^o or that of $\nabla\phi^o$ is out of phase by 26.5% of the period T and that of $\boldsymbol{\psi}^o$ by 38.1%. This phenomenon is in agreement with the principle of causality: the effects appear

successively on the vertices of the primal contour, then on the vector potential of the dual contour. The dual-flow retro-acts on the primal flow, the two accelerations $-\nabla\phi^o$ and $\nabla^d \times \boldsymbol{\psi}^o$ are thus dynamically entangled. The scalar potential ϕ^o and its gradient $-\nabla\phi^o$ are in phase, while the vector potential $\boldsymbol{\psi}^o$ and its dual curl $\nabla^d \times \boldsymbol{\psi}^o$ are in phase opposition.

Let us now consider an overall rotation around the unit vector $\mathbf{n}$ imposed by the accumulation potential $\boldsymbol{\psi}^o = 1$ kept constant over time; the velocity on the segments is being free to evolve to ensure the mechanical balance satisfied by the equation of motion. The resolution of this problem for the geometric system in Fig. 2 leads to a velocity which is a linear function in time of the form $\boldsymbol{v} = 2t\mathbf{t}$, corresponding to a rotational movement around $\mathbf{n}$ such that the barycentric velocity is zero. The scalar potential is also zero on the 4 vertices of the primal topology. Consequently, the terms of inertia are also zero. Mechanical equilibrium is then described by the equality $\partial\boldsymbol{v}/\partial t = \nabla^d \times \boldsymbol{\psi}^o$ where the last term is the nonzero dual curl of $\boldsymbol{\psi}^o$.

If the vector potential is variable in the form $\boldsymbol{\psi}^o = \cos(\omega t)$, then the velocity components are also periodic over time, $\boldsymbol{v} = (1/\omega)\ \sin(\omega t)\ \mathbf{t}$, but the scalar potential ϕ remains identically zero throughout the simulation as the divergence of the velocity. In this case the effects of shearing and of rotation lead to an incompressible block rotation motion; again Eq. (9) is satisfied.

Except for the very weak time constants, the laws of discrete mechanics give the same results as those of continuum mechanics. The physical model is, however, noticeably different: its geometrical nature makes it possible to understand the powerful and complex interactions of the two major actions of mechanics, compression and rotation.

3.2 Nonlinear pendulum

The case of a simple pendulum consisting of a mass m (called bob named by $\mathcal{B}$) suspended from a fixed axis by a rigid rod of length r is in static equilibrium when the downward gravity is aligned with the rod. When the mass is moved away from its equilibrium position by an angle θ_0 it is subjected to gravity and inertia in a periodic planar motion of constant amplitude if the friction is null. Friction in viscous air can be neglected at least at first.

The application of the second law of dynamics $\mathbf{F} = m\,\boldsymbol{\gamma}$ allows us to model very simply the motion of the pendulum when gravity is constant and to find the differential equation:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{r}\ \sin\theta = 0 \tag{14}$$

where $\theta(t)$ is the angle defined by the rod axis and the vertical direction.

Another method is to use the angular momentum. In Newton's second law, force is equal to the mass times the acceleration and torque obeys a similar law. When we replace torque with force and rotational inertia with mass and angular acceleration with linear acceleration, we get Newton's second law back out. In fact, this equation is Newton's second law applied to a system of particles in rotation around a given axis. This method leads to the same result (14) as obtained through force analysis.

Equation (14) has no simple analytical solution except for small angles θ_0 for which the sine can be assimilated to its angle, $\sin\theta \approx \theta$. The solution is then $\theta(t) = \theta_0\ \cos\left(\sqrt{g/r}\ t\right)$ and the period $T = 2\,\pi\,\sqrt{r/g}$. Approximate formulas can be found in the literature [16] as the angle increases.

Lagrangian mechanics is a very efficient method to solve many problems in solid mechanics. In the general case the Lagrangian is written:

$$\mathcal{L} = T - U \tag{15}$$

where T is the total kinetic energy and U is the total potential energy of a mechanical system. The equation of motion is then written:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right) - \frac{\partial\mathcal{L}}{\partial q_i} = F_i \tag{16}$$

where q signifies generalized coordinates and F signifies nonconservative forces acting on the mechanical system null in the case of the simple pendulum.

By expressing the total kinetic energy T from the velocity $\boldsymbol{v}$ of the pendulum and potential energy as a function of gravity we easily find differential equation (14). For angles θ_0 larger than those where the small

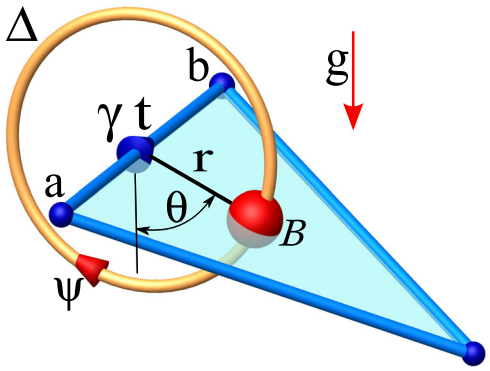

Fig. 4 Simple pendulum, a bob $\mathcal{B}$ linked by a rigid rod to a fixed point, in a constant and uniform gravitational field $\boldsymbol{g}$; the primal surface $\mathcal{S}$ composed of segments such that $\Gamma = [a, b]$ of unit vector $\mathbf{t}$ and the dual contour Δ oriented along $\mathbf{n}$ is orthogonal to the primal surface, $\mathbf{n} \cdot \mathbf{t} = 0$

angle approximation is justified, the solution of the problem is more difficult to evaluate. It takes the form of an incomplete elliptic integral which can be approximated by a Legendre polynomial, a power series of θ or a Fourier series. These solutions can be found in many textbooks.

This problem is now taken up again by implementing discrete mechanics. The basic principle of the law of discrete mechanics is based on the concepts of electromagnetism, which significantly modifies the modeling of the phenomena involved in a mechanical system. In the case of the simple pendulum where a body $\mathcal{B}$ rotates around the axis Γ, this motion generates an angular momentum which becomes, in discrete mechanics, an angular acceleration $\boldsymbol{\gamma}$. In general, the acceleration of a particle or a material medium is the sum of the translational acceleration and the angular acceleration; in this case, only the angular acceleration carried by the segment $\boldsymbol{\gamma}$ remains. The two contours of the geometric structures shown in Fig. 1 are crossed by direct and induced velocities $\boldsymbol{v}$ which are dynamically interleaved. For the chosen example, the velocity $\boldsymbol{v}$ is that of the dual contour pendulum, but the angular acceleration remains represented on the segment Γ.

The point of view adopted here is the direct use of Eq. (11) without trying to model again the phenomena involved in this problem. Indeed, the modeling of all the mechanical phenomena has already been done beforehand to derive this Eq. [5]. It is supposed to integrate all the present mechanisms of fluid flows, stresses and displacements of solids or linear/nonlinear wave propagation. The objective is to reduce this equation according to the problem posed by removing certain terms, for example by removing the compression terms represented by the gradient term of the scalar potential when the phenomenon is naturally incompressible. The source terms $\boldsymbol{h}_s$ must however be adjusted to take into account external accelerations, for example gravitational, capillary or magnetic effects. These will always be written in two terms of a Helmholtz–Hodge decomposition. Once the equation of motion has been reduced on the basis of physical considerations related to the problem at hand, it can be solved analytically if it is in the form of a differential equation whose solutions are known. In other more complex cases, the solution must be sought in an approximate manner using a computational method. In the case of the pendulum, the compression effects are zero and all terms of Eq. (11) defined by gradients are eliminated. All that remains is the time derivative $\partial \boldsymbol{v}/\partial t$ of the pendulum velocity, the dual rotation terms of the vector potential $\nabla^d \times \boldsymbol{\psi}$ and, of course, the source term.

From a physical point of view, the pendulum is subjected to two equal accelerations, (i) the inertial acceleration which restores the energy accumulated over time to ensure the oscillating motion and (ii) the acceleration of gravity which tends to pull the pendulum in the direction of gravity. Without the presence of viscous effects, the pendulum swings indefinitely. The inertia terms, of course, ensure the instantaneous mechanical balance between the potential energy and the kinetic energy of rotation. Once the compression terms are removed, the velocity $\boldsymbol{v}$ corresponds to the rotation of the pendulum around the axis. In electromagnetism, a direct current on the Γ segment generates an induced current in the Δ circuit and the opposite phenomenon also exists. In this case, the circulation of a rotational motion associated with the pendulum does not generate a translational motion on the Γ axis and the motion remains in the plane of rotation of the pendulum. The total acceleration is reduced to the angular acceleration; the equivalent of angular momentum in classical mechanics is carried by the Γ axis.

Consider the primal and dual structures of Fig. 4. The bob is placed at the barycenter of the primal surface $\mathcal{S}$ at a distance r from the segment Γ of orientation $\mathbf{t}$ making an angle θ with the vertical direction. With the

same notations as introduced above, the acceleration $\boldsymbol{\gamma}$ and the velocity $\boldsymbol{v}$ are associated with this segment of extremities a and b. The potential vector $\boldsymbol{\psi}$ is orthogonal to the facet $\mathcal{S}$ and directed by the normal $\mathbf{n}$ to it. The vertical gravity $\mathbf{g}$ is assumed to be uniform and constant. The primal geometric structure is intimately associated with the bob, and the primal surface is thus attached to the pendulum motion. It is to be noted that the mass m of the pendulum is not considered as it is for all the cases treated in the framework of discrete mechanics.

The gravitational contribution to the acceleration of a particle or a material medium $\boldsymbol{h}_g$ is defined by $\phi_g = \mathcal{G}\, M/r$ where $\mathcal{G}$ is the universal gravitational constant, M is the mass of the body, and r is the distance between the point considered and the center of gravity of the body. Assuming that the experiment takes place on the surface of the Earth, we find the classical value of vertical gravity $\mathbf{g} = -9.81\,\mathbf{e}_z$. In classical mechanics, only the gravity vector is expressed by a scalar potential $\nabla\phi_g$. In discrete mechanics, gravity is not only derived from a scalar potential but also from a vector potential $\boldsymbol{\psi}_g$:

$$\mathbf{g} = -\nabla\phi_g + \nabla^d \times \boldsymbol{\psi}_g \tag{17}$$

where gravity has a zero component on Γ and a nonzero component which corresponds to $\boldsymbol{\psi}_g = g\, r/2\, \sin\theta\, \mathbf{n}$ whose dual curl $\nabla^d \times \boldsymbol{\psi}_g = -g\, \sin\theta\, \mathbf{t}$ reproduces the result on $\mathbf{t}$. The discrete equation of motion is written as follows:

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \nabla^d \times \left(\boldsymbol{\psi}^o - J_\Delta\, \nabla \times \boldsymbol{v} + \boldsymbol{\psi}_g\right) \tag{18}$$

where $\boldsymbol{\psi}^o$ is the potential vector of mechanical equilibrium corresponding to the potential energy of the bob at a given moment, defined to within a constant from its initial position. The quantity J_Δ is the moment of inertia per unit mass and time; this term replaces $\mathrm{d}t\, c_t^2$ of the general formulation. As mass characterizes translational energy, the moment of inertia characterizes inertia with respect to rotational motion; there is formally an equivalence between the moment of inertia and the rotational energy expressed by the potential vector $\boldsymbol{\psi}$. The notion of equivalence between quantities of different nature is justified by the same form of the equation to which they are solutions. Indeed, the bob is solid and its rotational motion is rigid which induces a constant primal curl, $\nabla \times \boldsymbol{v}$ constant and its dual curl zero, $\nabla^d \times (\nabla \times \boldsymbol{v}) = 0$. Equation (18) can be simplified in the form:

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \nabla^d \times \left(\boldsymbol{\psi}^o + \boldsymbol{\psi}_g\right) \tag{19}$$

Exchanges between the available potential energy represented by $\boldsymbol{\psi}^o$ and the gravitational energy defined by $\boldsymbol{\psi}_g$ lead to an accelerated periodic motion. The angular acceleration $\boldsymbol{\gamma} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t$ carried by the segment Γ is a discrete representation of the angular momentum $\mathbf{L}$, but these two quantities are not expressed with the same units and are derived from different formalisms. The rotational velocity $\boldsymbol{v}$ on the dual contour Δ can be expressed from the radius r and the angular velocity as:

$$\boldsymbol{v} = r\, \frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{20}$$

and its material derivative in time is the angular acceleration $\mathrm{d}^2\theta/\mathrm{d}t^2$ on the contour Δ. The change of unknown results in a higher order of the differential equation and thus in an additional integration constant. In fact θ_0, the chosen initial angle, replaces the vector potential $\boldsymbol{\psi}^o$ of Eq. (18) at the initial time; these two quantities are then completely related to the potential energy at any time. As the radius r is constant, vector equation (19) reduces to the simple differential equation of the pendulum:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{g}{r}\, \sin\theta \tag{21}$$

This equation admits solutions in the form of series of functions whose convergence can be slow. It is always possible to transform it into a differential system:

$$\begin{cases} \dfrac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -g\, \sin\theta \\[2ex] \dfrac{\mathrm{d}\theta}{\mathrm{d}t} = \dfrac{\boldsymbol{v}}{r} \end{cases} \tag{22}$$

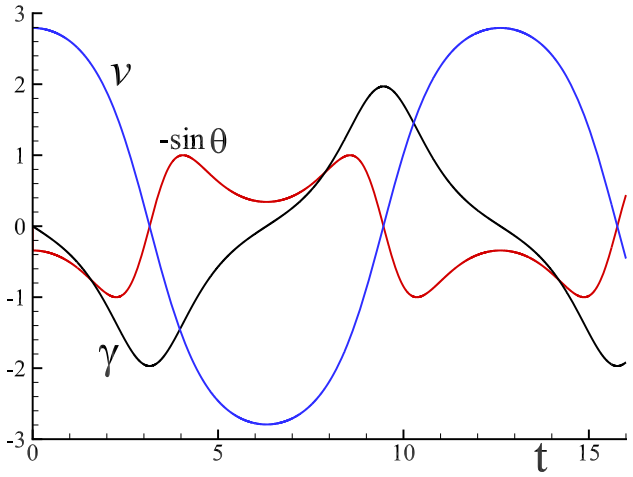

Fig. 5 Pendulum initially positioned at angle $\theta_0 = 160°$; the evolutions with time of acceleration $\boldsymbol{\gamma}(t)$, velocity $\boldsymbol{v}(t)$ and angle $\theta(t)$ show a periodic oscillating behavior

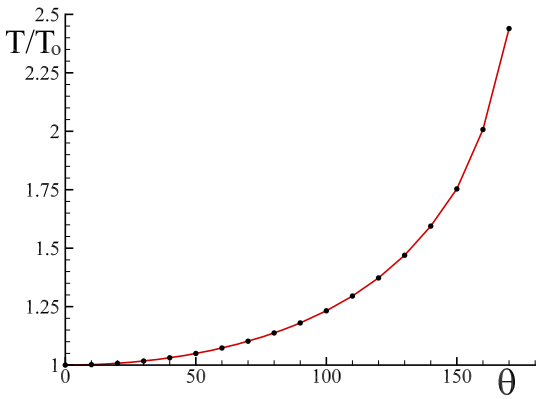

Fig. 6 Deviation of the true period of a pendulum from the small-angle approximation of the period

The solution of system (22) is realized using a Runge–Kutta time scheme of order two or four. Figure 5 shows the solution obtained with the fourth-order Runge–Kutta scheme for an initial angle $\theta_0 = 160°$.

If T_0 is the period corresponding to $\theta_0 \to 0$, the ratio T/T_0 increases slowly and then diverges when $\theta_0 = 180°$ which corresponds to an unstable state of the system. The evolution of the relative period is given in Fig. 6. The values obtained are very precisely those obtained by different authors using incomplete elliptic integrals.

The discrete formulation can be used while taking into account viscous effects by introducing ν an equivalent kinematic viscosity localized at the barycenter of the primal surface $\mathcal{S}$. Under these conditions the equation of motion becomes:

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \nabla^d \times \left(\boldsymbol{\psi}^o - \nu\, \nabla \times \boldsymbol{v} + \boldsymbol{\psi}_g\right) \tag{23}$$

This added term corresponds exactly to that present when the discrete equation of motion is used for viscous fluids where the instantaneous shear stress is expressed using the vector potential $\boldsymbol{\psi}^o = -\nu\, \nabla \times \boldsymbol{v}$. Figure 7 illustrates qualitatively the influence of the effect of the viscosity of air on the pendulum which reduces the amplitude of the oscillations with time.

Equation (23) is a model of the pendulum oscillating in air. A more rigorous simulation could of course be done from system (9) taking into account the pressure effects on the moving sphere. Of course the fluid current flowing on the primal Γ contour is zero because the flux induced by the periodic motion of the pendulum is damped in the air; indeed transverse waves in fluids are almost instantaneously dissipated and the attenuation factor α_t is very close to unity. However, the physical model is still legitimate to represent the direct and

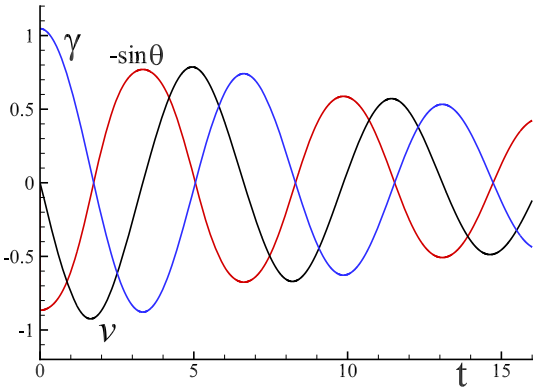

Fig. 7 Pendulum initially positioned at angle $\theta_0 = 60°$; evolution with time of acceleration $\boldsymbol{\gamma}(t)$, velocity $\boldsymbol{v}(t()$ and angle $\theta(t)$ when viscous effects are taken into account by the vector potential $\boldsymbol{\psi}^o = -\nu\, \nabla \times \boldsymbol{v}$

induced currents in the primal and dual structures which can switch their role. In the case of electromagnetic waves the entanglement would be translated by the longitudinal and transverse velocities equal to the speed of light without attenuation. In this case, the fictitious current in the primary loop allows to model the effect of a viscous friction by the law $-\nu\, \nabla \times \boldsymbol{v}$.

The simple pendulum problem has been solved from discrete equation (9) and, contrary to classical approaches using Newton's second law or Lagrangian mechanics, it does not require any additional modeling. Indeed, the laws of classical mechanics generally form systems of equations including the momentum balance, the energy balance, the conservation of mass, the constitutive laws associated with the boundary conditions. This is the case of the Navier–Stokes equation which includes the three components of the equation of motion associated with the law of conservation of mass and possibly conservation of energy. To close the system, the set must contain as many equations as variables, which justifies the integration of additional constitutive laws as a state law. This splitting of the balances and the overabundance of the physical quantities used can lead to inconsistencies preventing the strict conservation of the total energy.

The discrete law of motion is autonomous; the only variable in the equation is the velocity. The potentials ϕ^o and $\boldsymbol{\psi}^o$ are updated at each time step from the divergence and the curl of the velocity. The physical characteristics are defined elsewhere, independently of the modeling of the phenomena. The boundary conditions are integrated in the law of motion itself; this one is both a law of conservation of total energy and also of acceleration. Galileo's equivalence principle expressed in terms of acceleration and the equivalence principle of mass and energy of special relativity give it properties of invariance and symmetries which are those defined by Noether's theorem. Indeed, the discrete equation of motion is the sum of two Lagrangians, each including potential and kinetic energy terms, respectively, for the conservation of translational and rotational energy. Its general form is a nonlinear wave equation, which gives it the status of a relativistic equation.

The equation of motion expresses intrinsically that the inertial acceleration is equal to the gravitational acceleration. Of course the proposed approach is not as intuitive as the classical approaches of Newtonian or Lagrangian mechanics where the space vectors are represented by their components in a reference frame with three preferred directions. In discrete mechanics the components are only defined on segments oriented in a local reference frame. For example the divergence-free component of inertia for a two-dimensional domain is not expressed in this plane. In spite of this lack of simplicity, discrete mechanics allows to find the main unquestionable results of classical mechanics.

3.3 Uniform acceleration of an isolated particle

The case of a single particle with or without mass subjected to an external acceleration $\boldsymbol{h}_s = g\, \mathbf{e}_x = \nabla \phi_s$ from a state of rest is of a different nature, as the movement is not in mechanical equilibrium. Consider a particle at rest at the initial time, set in motion by means of an acceleration imposed from the outside, a gravitational field or a magnetic field, to the charged particles. In a Lagrangian description, movement is described by the time-dependent position of the particle $x(t)$ and velocity $u(x(t), t) = \mathrm{d}x/\mathrm{d}t$. The equation of discrete motion

(11) to the rectilinear movement of the particle from its origin located from at $x = 0$ from $t = 0$ is rewritten:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\phi^o - \mathrm{d}t\, u^2\, \frac{\mathrm{d}u}{\mathrm{d}x}\right) + g \tag{24}$$

The difference with Eq. (3) lies within the compression term; in fact, nothing suggests that the wave upstream and downstream is already in a state of equilibrium. The particle whose instantaneous velocity is equal to $u(t)$ compresses the downstream wave in order to reduce the divergence of the velocity until it is zero. The compression energy is equal to $\phi^o(t)$. The velocity of the particle tends toward the celerity of the medium $u \to c$, and the second member then becomes zero. The physical interpretation described by the equation of motion (3) becomes clearer: the source term $\boldsymbol{h}_s$, which is constant, induces an acceleration of the particle or of the material medium in the direction of this vector; the velocity increases and the energy accumulates over time in the equilibrium potential ϕ. Then, the divergence of the velocity tends toward zero and the mechanical equilibrium is reached when the acceleration becomes zero; we thus have:

$$-\nabla\phi + \boldsymbol{h}_s = 0 \tag{25}$$

The particle continues its rectilinear movement at a constant velocity equal to its celerity. The principle of relativity intrinsic to the equation of discrete motion allows the particle to keep the velocity acquired. In order to quantify the movement of the particle in terms of velocity and position in a more general vision, let us pose the following quantities to resize distance, time, velocity and energy per unit of mass:

$$x^* = \frac{x\, g}{c^2}; \quad t^* = \frac{t\, g}{c}; \quad u^* = \frac{u}{c}; \quad \phi^{*o} = \frac{\phi^o}{c^2} \tag{26}$$

where the starred quantities are the dimensionless variables. To lighten the writing, the same symbols are used to express the dimensionless quantities. Equation (24) thus becomes:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\phi^o - \mathrm{d}t\, u^2\, \frac{\mathrm{d}u}{\mathrm{d}x}\right) + 1 \tag{27}$$

The compression energy ϕ^o is that accumulated at time t from the initial time:

$$\phi^o(t) = -\int_0^t u^2\, \frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}\tau = -\int_0^t \frac{\mathrm{d}t}{\mathrm{d}x}\, u^2\, \mathrm{d}u = -\frac{u^2}{2} \tag{28}$$

Thus, at each instant, the compression energy of the wave is exactly equal to the kinetic energy acquired by the particle. As $g\, \mathbf{e}_x = \nabla(g\, x)$ in real variables, the scalar potential of the acceleration becomes equal to:

$$\phi(t) = x - \phi^o(t) = x - \frac{u^2}{2} \tag{29}$$

Equation (27) becomes:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u^2}{2}\right) \tag{30}$$

The position of the particle x and time t is not independent, and relation (30) is not an equation but an identity; each member expresses the material derivative in a Lagrangian description, because $\mathrm{d}x/\mathrm{d}t$ is none other than the velocity u. In fact the equation of discrete motion is not a simple differential equation. It involves a temporal process called "accumulation" where the description of the mechanical equilibrium at the instant t is deduced from that at the previous equilibrium instant t^o by a Lagrangian formulation composed of an equation on acceleration and its updates on velocity, energy and spatial coordinates:

$$\begin{cases} \boldsymbol{\gamma} = 1 - \dfrac{1}{u}\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\phi^o - u\, \dfrac{\mathrm{d}u}{\mathrm{d}t}\right) \\ \phi^o - u\, \dfrac{\mathrm{d}u}{\mathrm{d}t} \longmapsto \phi^o \\ u^o + \mathrm{d}t\, \boldsymbol{\gamma} \longmapsto u^o \\ x^o + \mathrm{d}t\, u \longmapsto x^o \end{cases} \tag{31}$$

This system is autonomous, it does not require a priori knowledge of the position of the particle as a function of time, nor recourse to one of the Lorentz transformations. This problem of the evolution of the uniformly accelerated particle requires only knowledge of the initial conditions (u^o, x^o, ϕ^o), zero for a particle at rest at the initial time. While the resolution of the Eulerian version given by system (9) is very robust, Lagrangian version (31) has a term in $1/u$ in the equation which makes it substantially more complicated to obtain the solution $(u(t), \phi(t), x(t))$. These numerical difficulties in Lagrange variables lead, however, to a behavior of the expected solution, with velocity naturally tending toward a constant value over time. It is possible to find a more robust alternative by considering the energy necessary for the particle to overcome inertia and a velocity which tends toward celerity. This energy, by definition equal to $c^2/2$, is expressed as a function of the integral of the acceleration from zero to infinity, i.e., in dimensionless form:

$$\phi_c = \int_0^\infty \boldsymbol{\gamma}(x)\, \mathrm{d}x = \frac{1}{2} \tag{32}$$

Acceleration is thus a decreasing function of x in $\mathcal{O}(x^{-n})$, whereby $n = 3$ in order to satisfy condition (32). The initial condition corresponding to an acceleration equal to unity for $x = 0$, satisfied by equation system (31), leads to the following form:

$$\boldsymbol{\gamma} = \frac{1}{(1+x)^3} \tag{33}$$

The imposed acceleration $\boldsymbol{h}_s = 1$ is transformed partly into compression energy equal to ϕ^o. This compression energy is transferred to the particle to increase its own kinetic energy ϕ_c and therefore its own acceleration $\boldsymbol{\gamma}$. At each instant we have $\phi_c = \phi^o$ and the vector potential of the acceleration is equal to $\phi = x - \phi^o$; equilibrium is reached when $-\nabla\phi + 1 = 0$. The imposed acceleration $\boldsymbol{h}_s$ then no longer acts on the particle, the movement is incompressible, and the particle moves at a velocity equal to the celerity $u = c$.

The solution to the problem in dimensionless form obtained by incremental system (31) corresponds as expected to the solution obtained in the context of special relativity.

$$\begin{cases} \boldsymbol{\gamma}(t) = \dfrac{1}{(1+x)^3} = \dfrac{1}{(1+t^2)^{3/2}} \\ u(t) = \dfrac{\sqrt{x(2+x)}}{(1+x)} = \dfrac{t}{\sqrt{1+t^2}} \\ \phi(t) = x - \dfrac{u^2}{2} \end{cases} \tag{34}$$

Figure 8 shows the evolutions obtained numerically by system (31) and acceleration (33) over time: (i) acceleration $\boldsymbol{\gamma}$, (ii) velocity $u(t)$, (iii) scalar potential $\phi(t)$ and (iv) the coordinate $x(t)$, (v), the divergence $\delta = \nabla \cdot \boldsymbol{v}$ shows that the movement becomes incompressible, explaining that the velocity of the particle tends toward celerity.

At short times, the acceleration is equal to unity, the velocity increases in the form $u(t) \approx t^2/2$ and the divergence of the velocity is then in $\mathcal{O}(t^{-1})$; at this point the accumulated energy is very close to zero. When the time increases beyond $t \approx 0.5$ the potential $\phi = x - \phi^o$ increases, the natural acceleration $\boldsymbol{\gamma}$ decreases and the growth in velocity slows down. The acceleration then tends toward zero and the velocity toward the celerity of the medium. The root cause of this limitation is the divergence of the motion of the particle, which goes from $\mathcal{O}(t^{-1})$ for Newtonian mechanics to $\mathcal{O}(t^{-3})$ in relativity and in discrete mechanics when time increases.

It is possible to compare the approach of special relativity (S.R.) to that of discrete mechanics (D.M.). They can be synthesized in the formal writing of the equations by returning to the real variables:

$$\begin{cases} S.R.: & \dfrac{\mathrm{d}(m\,u)}{\mathrm{d}t} = m_0\, g \\ D.M.: & \dfrac{\mathrm{d}u}{\mathrm{d}t} = -\nabla\phi + g \end{cases} \tag{35}$$

where the mass in motion is equal to $m = m_0\,\gamma$ with $\gamma = 1/\sqrt{1 - u^2/c^2}$ the Lorentz factor and m_0 the mass at rest.

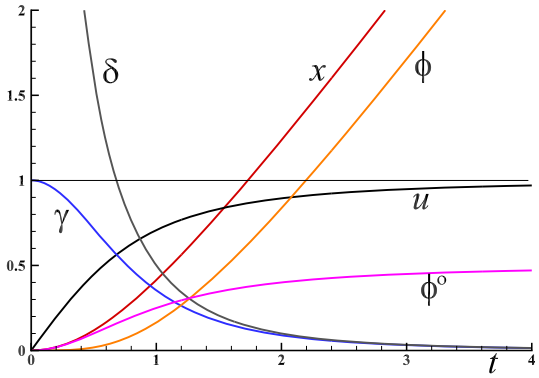

Fig. 8 Acceleration of an isolated particle with or without mass over time: $u(t)$ its velocity, $x(t)$ its position, $\boldsymbol{\gamma}(t)$, its acceleration and $\delta(t) = \nabla \cdot \boldsymbol{v}$ the divergence of the velocity. The scalar potential is equal to $\phi(t) = x - u^2/2$ and kinetic energy to $\phi^o(t) = u^2/2$

While the result is identical, the physical modeling of the phenomenon is not at all the same. First of all, equation (35) of special relativity is a transposition of Newton's second law, while discrete mechanics translates a conservation of accelerations. Thus the mass can vary, but it does not tend a priori toward infinity, while the velocity tends toward the celerity of the medium (material medium or vacuum). Besides, the material velocity or that of the particle is a notion which is disjointed from that of celerity, and the ratio u/c of the Lorentz transformation is not self-evident, except the consideration as a principle that velocity is always limited by celerity. In relativity, the velocity of the photon cannot exceed the celerity of light in a vacuum in principle; this is not the case in discrete mechanics where the velocity of the photon is not limited, and if it tends toward celerity, this corresponds to a decrease in the sum of the accelerations which are applied to it. In the problem presented, this is indeed the case because the trajectory does not have any curvature.

The equation of motion of discrete mechanics does not present itself as a relation whose solution we seek directly; it corresponds at each instant to an accumulation of the instantaneous compression energy $\mathrm{d}t\, u^2\, \nabla \cdot \boldsymbol{v}$ within the scalar potential ϕ^o. Thus, for short times, the solution is equal to $u(t) \approx g\, t$; then, the term $\nabla\phi$ increases and causes the reduction in the proper acceleration $\boldsymbol{\gamma}$. Gradually the velocity stabilizes toward the celerity of the medium.

The case of the zero mass photon is of particular interest. In discrete mechanics, if it has no mass, it has kinetic energy per unit of mass, like massive particles, and its velocity increases gradually; this is not a special case. In relativity, the photon is assumed to always have a velocity equal to the celerity of light because it is already emitted by the source at this velocity. For charged particles, similarity parameters (28) make it possible to estimate the time constant $\tau \approx c/g$ (independent of mass) so that its velocity is close to the celerity, hence motivating the very high energies used in particle accelerators.

The potential ϕ^o is the energy per unit of mass corresponding to the compression effects. In the case where other gradient effects are imposed by accelerations, the energy is also accumulated in the scalar potential ϕ of the acceleration. The elementary energy corresponding to the compression present in the equation of motion (9), $\mathrm{d}t\, c^2\, \nabla \cdot \boldsymbol{v}$ implicates it in the variable velocity in a similar way to the rotation-shear term. The energy ϕ^o is then updated explicitly from the divergence of the velocity.

Analysis of the behavior of the discrete motion equation reveals the physical mechanism of the limitation of velocity to the celerity of the medium. The particle's own acceleration is equal to the sum of the accelerations: that which is imposed, **g**, of positive sign, and that of the opposite sign which translates the compression of the downstream wave until the divergence velocity is zero. The contraction of lengths and the dilation of time are absolutely necessary concepts in special relativity; they are induced by the initial choice of remaining within the framework of Newton's second law and of associating the mass in motion with the relationship between velocity and celerity. When the velocity increases, the distances contract and the time intervals expand. In discrete mechanics, the length and the time are absolute quantities which do not depend on the velocity. For example, time flows linearly in a discrete way by intervals $\mathrm{d}t$. When the velocity increases, it is the divergence which is reduced and which makes the wave less compressible, until it becomes incompressible when the velocity tends toward celerity. The abstractions of the theory of relativity are replaced here by simple concepts of mechanics, in particular the mechanics of compressible fluids.

The velocity is also limited in other similar situations in a continuous medium where a gas particle undergoes an expansion. This is the case of the shock tube where a gas is compressed upstream of a cap, while a vacuum or a reduced pressure is maintained downstream. At the initial moment the seal is broken and the gas undergoes an expansion upstream and a compression with a shock downstream. We observe that the gas velocity cannot be higher than the local sound celerity. This limitation is specific to rectilinear movements; the sound celerity can only be exceeded in the presence of curvatures, for example with a Laval nozzle or with adapted wing profiles.

When the particle is accelerated in a direction that is not aligned with its trajectory, it is no longer straight, and it presents a deviation that is the sign of a rotational action. The two terms of inertia are then in competition and the particle will be all the more deviated from a rectilinear trajectory that the external acceleration imposed will be important. The case of a black hole in astrophysics, defined by the Schwarzschild horizon, the radius below which light is trapped, is an emblematic example. Discrete mechanics simply obtains this same result by comparing the two inertia terms to the curvature of the gravitational potential.

4 Turbulence in fluids

The problem of turbulence in fluids is probably the one where entanglement plays the greatest role; by definition a turbulent flow is unsteady, this is the condition for the two terms of the equation of motion to be intertwined. The characteristics of turbulence are well known since Richardson and Kolmogorov, but the problem is not closed for all that; there are still questions about the phenomenology of vortex-stretching not to mention the mathematical problem of the existence of long-term solutions in three space dimensions. The question posed concerns only the potentiality of the discrete formulation to represent, in the same way as the Navier–Stokes equations, turbulent flows by direct numerical simulations.

4.1 Entanglement of compression and rotation effects in fluids

The strong coupling between all the terms of the discrete equation of motion is essential for its physical representativity. Indeed, the application of the divergence or primal curl operators on the equation of motion, for example the Navier–Stokes equation, modifies its meaning. The divergence operator is used in the time-splitting methods to project the equation on a zero divergence field, and the application of the primal curl operator leads to the elimination of the pressure term. These procedures break the entanglement between the compression and rotation terms. In most cases this is of little importance, especially in the steady state, but the treatment of turbulence becomes problematic when operators are used to transform the equation of motion. The incompressibility constraint $\nabla \cdot \boldsymbol{v} = 0$ is then imposed without any possibility of transfer by compression waves.

For example the application of the primal curl to the incompressible Navier–Stokes equation allows to eliminate the pressure gradient $-\nabla p$ to formulate the vorticity form:

$$\begin{cases} \dfrac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} - \boldsymbol{\omega} \cdot \nabla \boldsymbol{v} = \nu \, \nabla^2 \boldsymbol{\omega} \\ \nabla^2 \Psi = -\boldsymbol{\omega} \end{cases} \tag{36}$$

where $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ is the vorticity vector and Ψ is the potential velocity vector. Many methods use this formulation which has some advantages including the fact that it ensures the incompressibility constraint $\nabla \cdot \boldsymbol{v} = 0$.

However, formulation (36) inhibits any transformation between the compression and rotation energies. Incompressibility is a very restrictive assumption insofar as it propagates any pressure perturbation instantaneously to infinity. This quantity becomes a simple Lagrangian to ensure the mass conservation constraint under the incompressibility assumption.

The discrete primitive formulation represented by equation (37) has a remarkable property due to the Helmholtz–Hodge decomposition.

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\nabla \left(\phi^o + \phi_i - c_l^2 \, \mathrm{d}t \, \nabla \cdot \boldsymbol{v} \right) + \nabla^d \times \left(\boldsymbol{\psi}^o + \boldsymbol{\psi}_i - c_t^2 \, \mathrm{d}t \, \nabla \times \boldsymbol{v} \right) \tag{37}$$

The two terms of the right-hand side are locally orthogonal [4], but they are carried by the same Γ support. The compression energy represented by the first term cannot exchange directly with the last term which translates the rotation. This transfer is only possible if the motion is unsteady; the energy is thus redistributed toward one or the other term of the right-hand side. This is the remarkable idea of J.C. Maxwell's of combining magnetism and electrodynamics into electromagnetism. These two effects are dynamically intertwined.

The reasons for the breaking can be multiple; the splitting of the operators can be one of them. The Navier–Stokes equation translates the conservation of momentum, and the law of conservation of mass is attached to it; the Navier–Stokes equation does not ensure as such the conservation of mass. In an incompressible formulation, the pressure is deduced from this last law by assuming a link, through a state law, between the density and the pressure. Numerical methods such as the projection method [13] add an additional splitting.

It is quite different for Eq. (37), even for motions considered incompressible. Indeed the term $d\phi = -\mathrm{d}t\, c_l^2\, \nabla \cdot \boldsymbol{v}$ never becomes zero, when the celerity c_l increases it is the divergence that decreases. The quantity $d\phi$ is an energy per unit of mass transferred to the fluid by a compression process during the time lapse $\mathrm{d}t$. Dynamic entanglement makes sense in turbulent flows, where, by definition, velocity fluctuations have high frequency.

Law of discrete motion (9) expresses primarily the conservation of total energy per unit mass. The term on the left represents the change in total mechanical energy between a and b, and the two terms in the right-hand member represent the compression and rotation energies; each of the last two terms is a Lagrangian with a potential energy fixed by the potentials ϕ^o and $\boldsymbol{\psi}^o$, and a kinetic energy carried by the increments $\mathrm{d}\phi = \mathrm{d}t\, c_l^2\, \nabla \cdot \boldsymbol{v}$ and $d\boldsymbol{\psi} = \nu\, \nabla \times \boldsymbol{v}$. More precisely the equation of motion can be put in an integral form carried by the segment Γ:

$$\int_a^b \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \cdot \mathbf{t}\, \mathrm{d}l = -\int_a^b \nabla \left(\phi^o - \mathrm{d}t\, c_l^2\, \nabla \cdot \boldsymbol{v}\right) \cdot \mathbf{t}\, \mathrm{d}l + \int_a^b \nabla^d \times (-\nu\, \nabla \times \boldsymbol{v}) \cdot \mathbf{t}\, \mathrm{d}l \tag{38}$$

where $d\boldsymbol{v}/dt$ is the material derivative, $\nabla \left(\mathrm{d}t\, c_l^2\, \nabla \cdot \boldsymbol{v}\right)$ the acceleration which represents the compression fluctuation and finally $\nabla^d \times (\nu\, \nabla \times \boldsymbol{v})$ the acceleration associated with the rotation fluctuation; the dual-curl operator eliminates the rigid rotational motion.

In the Helmholtz–Hodge decomposition

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\nabla \left(\phi^o + \phi_i\right) + \nabla^d \times \left(\boldsymbol{\psi}^o + \boldsymbol{\psi}_i\right) \tag{39}$$

the total energy represented by the integral on a segment of the first member cannot be dissociated from one of the components, it is the total energy that is conserved, and this is the very object of the law of dynamics. At each instant the compression energy and the shear energy are exchanged only through the term in time $\partial v/\partial t$ because these two terms are orthogonal and exchange nothing directly.

This concept is called dynamic entanglement; it comes from the remarkable idea of J.C. Maxwell who founded electromagnetism by combining the laws of electrodynamics with those of magnetism. An alternating current in an electrical conductor produces an induced electric current of the same frequency in a circuit surrounding the primary conductor, if the current is direct the coupling does not exist; it is this observation that led to the success of alternating current for the distribution of electricity.

In mechanics, this concept translates into the circulation of a fluid in a circuit composed of Γ segments such as those forming the primary surface of Fig. 1. This circulation allows to calculate the primal curl to define the potential vector $\boldsymbol{\psi} = \nu\, \nabla \times \boldsymbol{v}$. In its turn the circulation of this dual vector reprojects a flow on the segment Γ. The two accelerations $-\nabla\phi$ and $\nabla^d \times \boldsymbol{\psi}$ circulate on the same segment without interacting; the two components are indeed orthogonal. On the other hand, one of these components modifies the velocity in time, which has the effect of affecting the other one *a posteriori*.

In turbulence this mechanism of transferring compression and rotation energies is essential. In classical mechanics the role of compression is underestimated if not neglected. But even for flows considered incompressible, it is not the fact that $\nabla \cdot \boldsymbol{v}$ is close to zero that is important, what really counts is the quantity $\mathrm{d}t\, c_l^2\, \nabla \cdot \boldsymbol{v}$ which remains defined when $\nabla \cdot \boldsymbol{v} = 0$ because the celerity c_l tends to infinity.

4.2 A local point of view

Continuum mechanics is part of the larger concept of continuous medium where any quantity is first defined on a bounded domain Ω limited by a surface Σ before being localized at a point from the divergence theorem.

Thus the momentum and mass balances are first established on the Ω domain, and then, the different quantities are assigned at a point. Let us consider the momentum equation associated with the conservation of mass:

$$\begin{cases} \rho \dfrac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \nabla \cdot \boldsymbol{\sigma} + \mathbf{f} \\ \dfrac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \, \nabla \cdot \boldsymbol{v} = 0 \end{cases} \tag{40}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor and $\mathbf{f}$ a source term per unit mass. Boundary conditions complete the local formulation.

The kinetic energy theorem obtained by a dot product of (40) by $\boldsymbol{v}$ and integrating over the volume can be interpreted as a kinetic energy balance equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \rho \, \frac{|\boldsymbol{v}|^2}{2} \, \mathrm{d}v = \int_{\Sigma} \boldsymbol{\sigma} \, \boldsymbol{v} \cdot \mathbf{n} \, \mathrm{d}s - \int_{\Omega} \boldsymbol{\sigma} : \mathbf{D} \, \mathrm{d}v + \int_{\Omega} \boldsymbol{v} \cdot \mathbf{f} \, \mathrm{d}v \tag{41}$$

where $\mathbf{D}$ is the strain rate tensor.

For a Newtonian fluid with constant physical properties such that $\boldsymbol{\sigma} = -p \, \mathbf{I} + \boldsymbol{\tau}$ where $\boldsymbol{\tau}$ is the viscous stress tensor and leaving aside the external forces, the kinetic energy theorem becomes:

$$\frac{d}{dt} \int_{\Omega} \rho \frac{|\boldsymbol{v}|^2}{2} \, \mathrm{d}v = - \int_{\Sigma} p \, \boldsymbol{v} \cdot \mathbf{n} \, \mathrm{d}v + \int_{\Omega} p \, \nabla \cdot \boldsymbol{v} \, \mathrm{d}v + \int_{\Sigma} \boldsymbol{v} \, \boldsymbol{\tau} \cdot \mathbf{n} \, \mathrm{d}s - \int_{\Omega} \boldsymbol{\tau} : \nabla v \, \mathrm{d}v \tag{42}$$

When the velocity is zero or $\boldsymbol{v} \cdot \mathbf{n} = 0$ at the edge the first integral of the second member disappears, the second integral also disappears if the flow is incompressible as well as the third if the normal stresses are zero at the edge; this is the case for the Taylor–Green vortex example. The kinetic energy depends only on the viscous dissipation term represented by the last integral. For an inviscid fluid the kinetic energy is conserved during the motion. The elimination of the pressure from kinetic energy equation (42) used in the same way for the solution of the Navier–Stokes equation by some variational formulations leads to weak solutions. See [22] for a relevant discussion on kinetic energy preservation for the Euler equations.

It is quite different in discrete mechanics; this formulation corresponds to a strong solution of the equation of motion. The solution is sought on each of the Γ segments of the primal structure. The conservation is expressed as all other accelerations on this segment. The continuous medium frame is completely abandoned and replaced by a local discrete frame. The classical Galilean frame of reference is replaced by the local frame of reference of Fig. 1. As $\boldsymbol{v}$ is the velocity component on Γ and $\nabla\phi$ is the restriction of the potential gradient (pressure), these two vectors are collinear.

$$\frac{\mathrm{d}E_k}{\mathrm{d}t} = -\boldsymbol{v} \cdot \nabla\phi - \boldsymbol{v} \cdot \nabla^d \times (\nu \, \nabla \times \boldsymbol{v}) \tag{43}$$

The first term of the second member $-\boldsymbol{v} \cdot \nabla\phi$ is not zero *a priori*, and its sign is indeterminate. The last term represents the viscosity effects whose contribution is definitely converted into heat; it is the viscous dissipation. It is possible to model this term from Darcy's model by introducing an effective permeability K to transform it into $-(\nu/K) \, |\boldsymbol{v}|^2$. In any case, since ν and K are positive quantities, this term tends to reduce the kinetic energy with time following an exponential evolution.

The turbulent statistic $-\mathrm{d}E_k/\mathrm{d}t$ called dissipation in continuum mechanics is a by-product of the numerical solution. It does not depend fundamentally on the way it is computed, on the volume in continuum mechanics or on a segment in discrete mechanics. However, it reveals that the kinetic energy calculated during the resolution has a significantly different evolution by the two approaches.

The local point of view adopted in discrete mechanics makes it possible to ignore long distance influences, in particular those of the boundary conditions. In flows with very small time constants $\mathrm{d}t$ like those of turbulence, the interactions are limited in space by the discrete horizon $\mathrm{d}h = c_l \, \mathrm{d}t$; a fluctuation of a quantity at a point is not instantaneously felt by the boundaries of the domain. The interactions from one segment to another are actualized from cause to effect through the scalar potential common to two segments. Of course the modeling of a viscous flow with a larger time constant forces to consider the transformation of the transverse propagation term $-\mathrm{d}t \, c_t^2 \, \nabla \times \boldsymbol{v}$ into a diffusion term $-\nu \, \nabla \times \boldsymbol{v}$. This assumption is justified by the very rapid decay of transverse waves into heat by viscous dissipation.

The role of pressure is largely neglected in some models of turbulence, which introduce a turbulent pressure term that is associated with viscosity. The notions of pressure and viscosity are strictly disjoint in discrete mechanics. The pressure (scalar potential) is also largely underestimated in direct simulations performed in incompressible or compressible formulations using constitutive laws. Turbulence is characterized by a very small-scale locality and violent phenomena.

4.3 Discrete kinetic energy theorem

It is necessary to go back to the differences in concepts between continuum mechanics and discrete mechanics and, in particular, to the notion of local reference frame. In continuum mechanics we can verify on many examples that the Lamb vector $-\boldsymbol{v} \times \nabla \times \boldsymbol{v}$ is a gradient in the plane (x, y); its primal rotational must therefore be zero $\boldsymbol{\omega} = \nabla \times \boldsymbol{v} = 0$. The application of the curl operator to the inertial terms of the Navier–Stokes equation from terms $\boldsymbol{v} \cdot \nabla \boldsymbol{v}$ equal to $\nabla \left(|\boldsymbol{v}|^2/2 \right) - \boldsymbol{v} \times \nabla \times \boldsymbol{v}$ leads to search for the curl of the Lamb vector, $\nabla \times (-\boldsymbol{v} \times \nabla \times \boldsymbol{v})$. By posing $\boldsymbol{u} = \boldsymbol{v}$ and $\boldsymbol{w} = \nabla \times \boldsymbol{v}$, the vector identity $\nabla \times (\boldsymbol{u} \times \boldsymbol{w}) = \boldsymbol{w} \cdot \nabla \boldsymbol{u} - \boldsymbol{w} \nabla \cdot \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{w} + \boldsymbol{u} \nabla \cdot \boldsymbol{w}$ is simplified by considering $\nabla \cdot \boldsymbol{v} = 0$ and $\nabla \cdot (\nabla \times \boldsymbol{v}) = 0$. The curl of Lamb vector becomes $\nabla \times (\boldsymbol{v} \times \nabla \times \boldsymbol{v}) = \nabla \times \boldsymbol{v} \cdot \nabla \boldsymbol{v} - \boldsymbol{v} \cdot \nabla \nabla \times \boldsymbol{v}$ then $\boldsymbol{\omega} \cdot \nabla \boldsymbol{v} - \boldsymbol{v} \cdot \nabla \boldsymbol{\omega}$ by putting $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ and:

$$\nabla \times \left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \right) = \nabla \times \left(\frac{\partial \boldsymbol{v}}{\partial t} \right) + \boldsymbol{v} \cdot \nabla \left(\nabla \times \boldsymbol{v} \right) - \nabla \times \boldsymbol{v} \cdot \nabla \boldsymbol{v} = \frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \boldsymbol{v} \tag{44}$$

where $\boldsymbol{v} \cdot \nabla \boldsymbol{\omega}$ represents the advection of the vortex by the velocity field. The term $\boldsymbol{\omega} \cdot \nabla \boldsymbol{v}$ is zero in two dimensions of space because the two components are orthogonal, namely:

$$\nabla \times \left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \right) = \frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{\omega} \tag{45}$$

The fact that the result is not the same in two and three dimensions of space leads to the question of the legitimacy of this last term, which owes its existence only to the fact that $\boldsymbol{v}$ is a vector of space. If this vector was projected in each space plane of the orthogonal trihedron to each direction (x, y, z), operator $\nabla \times \boldsymbol{v}$ would be the curl of this projected vector but also that the component on $\mathbf{n}$ of vector $\boldsymbol{v}$ of $\mathcal{R}^3$. For each of the planes, the term $\boldsymbol{\omega} \cdot \nabla \boldsymbol{v}$ would be zero and expression (45) would be the same whatever the dimension of space.

For an incompressible flow with constant viscosity we find the 3D Helmholtz equation:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \boldsymbol{v} = \nu \, \nabla^2 \boldsymbol{\omega} \tag{46}$$

neglecting the baroclinic term $\Pi = (1/\rho^2) \, \nabla \rho \times \nabla p$ provided that the fluid is considered barotropic.

The representation of the velocity is realized in discrete mechanics by considering only its components $\boldsymbol{v}$ on an oriented segment Γ; it has no reality in a three-dimensional space. The primal facet $\mathcal{S}$ being planar the components $\boldsymbol{v}$ are also in this plane, and its primal curl $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ is orthogonal to it, $\boldsymbol{v} \cdot \boldsymbol{\omega} = 0$. This characteristic is not found in classical mechanics where the velocity is a three-dimensional space vector. Thus the difference between 2D turbulence and 3D turbulence described in classical mechanics does not exist in discrete mechanics; even in two dimensions of space of the (x, y) plane the inertia term is expressed through a dual curl of a vector oriented in the direction orthogonal to the considered plane.

Moreover, dynamical entanglement can be inhibited by the adopted physical model; the approximation of incompressibility $\nabla \cdot \boldsymbol{v} = 0$, although legitimate, poses the problem of applying this constraint within the equation of motion. The term $\nabla \left(\lambda_v \, \nabla \cdot \boldsymbol{v} \right)$ cannot be ignored by setting $\nabla \cdot \boldsymbol{v} = 0$ because the coefficient of compressive viscosity has no precise value for fluids; moreover Stokes's hypothesis $\lambda_v = (\lambda + 2/3 \, \mu) = 0$ is erroneous, including monoatomic gas, [11,21]. The value of the volume viscosity does not matter in the Navier–Stokes equation anyway, [8]. In discrete mechanics it is the term $\mathrm{d}t \, c_l^2 \, \nabla \cdot \boldsymbol{v}$ which expresses the exchanges by compression. This very important term expresses the successive compression and relaxation within a turbulent flow. Even if the longitudinal velocity of the fluid is very large, this term remains of the same order of magnitude as the other terms in the equation; even if the approximation $c_l \to \infty$ is adopted, the acceleration due to the compression effects remains limited. In classical mechanics, the effects of rapid pressure fluctuations on turbulence are underestimated or ignored by the application of the incompressibility constraint.

For numerical simulations the constraint is applied through the adopted formalism, for example by introducing from the beginning the variable $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ or by projecting the solution on a space with zero divergence. This is the case of pseudo-spectral methods and splitting methods where the constraint is realized separately from the resolution of the equation of motion. The application of this constraint modifies the energy exchanges between the inertial, the compression and the viscous terms. The full compressible formulation does not bring any improvement because it introduces an additional splitting by the application of external constitutive laws for example a state law linking density, pressure and temperature. In discrete mechanics the constitutive laws are absent; this is the potentials that carry them, for example $\phi^o = p/\rho^\gamma$ for an isentropic evolution; the pressure and the density can be deduced from it a posteriori.

Another noticeable difference lies in the choice of turbulence indicators. The main ones are the kinetic energy E_k integrated on the domain:

$$E_k = \frac{1}{\rho_0[\Omega]} \int_\Omega \rho \, \frac{\boldsymbol{v} \cdot \boldsymbol{v}}{2} \, \mathrm{d}v \tag{47}$$

and its temporal evolution, $\epsilon = -\mathrm{d}E_k/\mathrm{d}t$, the dissipation rate. In discrete mechanics the density is dropped and kinetic energy is defined by unit mass. The temporal evolution of the enstrophy integrated on the domain is written:

$$\mathcal{E} = \frac{1}{\rho_0[\Omega]} \int_\Omega \rho \, \frac{\boldsymbol{\omega} \cdot \boldsymbol{\omega}}{2} \, \mathrm{d}v \tag{48}$$

The enstrophy is only a kinematic quantity not related to the shear stress. Other criteria defined *a posteriori* allow us to understand the mechanisms of scale transfer during the decay of the scales of turbulence. The principle of discrete turbulence analysis is very different; it is based on the equation of motion itself. Indeed, the mechanisms of turbulence in fluids are intimately linked to the energy transfers between the different contributions represented in the discrete equation of motion: The line integrals of the different accelerations of (38) are all energies assigned to the Γ segment. The total energy is thus strictly conserved over time; the different contributions are completely related to each other. These five quantities are the indicators of discrete turbulence.

It is important to specify the differences in the kinetic energy theorem between continuum mechanics and discrete mechanics. These differences are not so much related to the presence or absence of mass as to the fact that we consider the integration on an elementary volume specific to the notion of continuous medium. Let us take the Navier–Stokes equation multiplied by the velocity vector of space $\mathbf{V}$ and, to simplify the matter, let us consider the only contribution of the pressure and an incompressible flow. Let us consider an elementary volume Ω limited by an impermeable surface Σ; the conservation of kinetic energy is written as follows:

$$\int_\Omega \rho \, \frac{\mathrm{d}|\mathbf{V}|^2}{\mathrm{d}t} \, \mathrm{d}v = -\int_\Omega \mathbf{V} \cdot \nabla p \, \mathrm{d}v \tag{49}$$

The term $\mathbf{V} \cdot \nabla p$ can be transformed into the form $\mathbf{V} \cdot \nabla p = \nabla \cdot (p\,\mathbf{V}) - p\,\nabla \cdot \mathbf{V}$; the last term in (49) vanishes when incompressibility is taken into account and the equation becomes:

$$\int_\Omega \rho \, \frac{d|\mathbf{V}|^2}{\mathrm{d}t} \, \mathrm{d}v = -\int_\Sigma p\,\mathbf{V} \cdot \mathbf{n} \, \mathrm{d}s \tag{50}$$

and as the surface is impermeable, $\mathbf{V} \cdot \mathbf{n} = 0$, it comes:

$$\int_\Omega \frac{d|\mathbf{V}|^2}{\mathrm{d}t} \, \mathrm{d}v = 0 \tag{51}$$

This observation translated by (51) is perfectly legitimate if we stick to the equilibrium on the volume Ω. For a material point $\mathrm{d}E_k/\mathrm{d}t = 0$, the kinetic energy remains constant when we follow the medium in its motion; this local form is however questionable. In the presence of viscous forces, the form adopted by many authors [2,25,28] only takes into account the latter terms and the energy decay becomes:

$$-\frac{\mathrm{d}|\mathbf{V}|^2}{\mathrm{d}t} = \boldsymbol{\varepsilon} \tag{52}$$

considered as the evolution of the only dissipation $\boldsymbol{\varepsilon} = 2\,\nu\,\mathbf{S} : \mathbf{S}$ where $\mathbf{S}$ denotes the deviatoric part of the rate-of-strain tensor. The problem remains for compressible flows where the divergence is not zero.

The major objection to this conclusion lies in the notion of continuous medium itself; the use of a transformation of a weak integral formulation leads to a loss of information, it amounts to asserting that locally one imposes $\mathbf{V} \cdot \nabla p = 0$, but these two vectors are not necessarily orthogonal. In continuum mechanics, the kinetic energy theorem specifies that the sum of the forces applied to a material medium is equal to the variation of its kinetic energy.

Its transposition into discrete mechanics is immediate: the sum of the accelerations applied to a material medium is equal to the variation of the kinetic energy per unit of mass; this is noted $\boldsymbol{e}_k$. It is both a scalar attached to the vertices of the primal structure and a vector related to the Γ segment, $\boldsymbol{e}_k = 1/2\,(\boldsymbol{v} \cdot \boldsymbol{v})\,\mathbf{t}$ since the vector $\boldsymbol{v}$ is itself associated with the segment Γ. This is not a new law; it is deduced from the equation of motion. In discrete mechanics the equation of motion (9) is multiplied by $\boldsymbol{v}$ and integrated over the length of the segment:

$$\int_\Gamma \frac{1}{2}\frac{\mathrm{d}|\boldsymbol{v}|^2}{\mathrm{d}t} = -\int_\Gamma \boldsymbol{v} \cdot \nabla\phi + \int_\Gamma \boldsymbol{v} \cdot \nabla^d \times \boldsymbol{\psi} \tag{53}$$

The potential $\phi = \left(\phi^o - c_l^2\,\mathrm{d}t\,\nabla \cdot \boldsymbol{v}\right)$ is the compression or translational kinetic energy and $\boldsymbol{\psi} = \left(\boldsymbol{\psi}^o - c_t^2\,\mathrm{d}t\,\nabla \times \boldsymbol{v}\right)$ is the rotational energy or angular kinetic energy. Form (9) of the equation of motion includes from the start the balance of the sum of the forces but also the moments. The acceleration $\boldsymbol{\gamma}$ is not only related to the translational motion but also translates the conservation of angular momentum.

Contrary to classical mechanics where the vector $\mathbf{V}$ is not necessarily collinear with the pressure gradient vector ∇p and similarly to viscous forces, the discrete formulation removes any interpretation on the orientation of the terms in the kinetic energy equation. In fact $\boldsymbol{v}$ is well collinear to $\nabla\phi$ and to $\nabla^d \times \boldsymbol{\psi}$; the results of the scalar product allow us to consider that $-\boldsymbol{v} \cdot \nabla\phi$ and $\boldsymbol{v} \cdot \nabla^d \times \boldsymbol{\psi}$ are at the same time scalars on the oriented segment and vectors. The unknown of the equation of motion being $\boldsymbol{v}$, the local kinetic energy will be obtained simply by a scalar product $\boldsymbol{e}_k = 1/2(\boldsymbol{v} \cdot \boldsymbol{v}) = 1/2|\boldsymbol{v}|^2$. The quantity $\boldsymbol{e}_k$ considered as an average on a segment allows to define the discrete theorem of the local kinetic energy:

$$\frac{\mathrm{d}\boldsymbol{e}_k}{\mathrm{d}t} = -\boldsymbol{v} \cdot \nabla\phi + \boldsymbol{v} \cdot \nabla^d \times \boldsymbol{\psi} \tag{54}$$

It is possible to define an average value E_k on the whole physical domain of the kinetic energy per unit of mass from its value on each segment of the primal structure $\boldsymbol{e}_k$:

$$E_k = \frac{1}{[\Gamma^*]} \int_{\Gamma^*} \frac{1}{2}\,|\boldsymbol{v}|^2\,\mathrm{d}l \tag{55}$$

where $[\Gamma^*]$ is the length measure of all segments. Similarly the global compression energy E_c is represented by an integral over the primal volume Ω:

$$E_c = \frac{1}{[\Omega]} \int_\Omega \phi^o\,\mathrm{d}v \tag{56}$$

Returning to local form (54), the velocity $\boldsymbol{v}$ is aligned not only with $\nabla\phi$ but also with $\nabla^d \times \boldsymbol{\psi}$. This last term corresponds to the rotation of the medium but only reflects dissipation in the case of a viscous fluid for which the potential vector is of the form $\boldsymbol{\psi} = \nu\,\nabla \times \boldsymbol{v}$; in this situation the transverse waves are completely attenuated over very small time constants ($\tau \approx 10^{-11} s$). The local variation of the kinetic energy of a material medium during its motion can thus be positive or negative, but the most important thing is that the integration to the whole volume Ω of the considered flow cannot be limited *a priori* to a monotonous decay. Indeed discrete law of motion (9) expresses the conservation of the total energy $\Phi = E_c + E_r$ where E_c is the compression energy per unit of mass and E_r the rotation energy. The kinetic energy E_k is only a part of the total energy. In the absence of viscous friction for a fluid, there remain two energies, E_c and E_k, the sum of which is indefinitely conserved over time in an Eulerian view. For a given flow, the velocity $\boldsymbol{v}$ and the potential ϕ are nonzero and fixed by the initial condition, the evolution in time of the system will be governed by only equation of motion (9); equation (54) is only a consequence of it. Like any mechanical system, the kinetic energy and the potential

energy will evolve in such a way as to preserve the total energy. The following form of the equation of motion allows us to be convinced of this:

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\nabla\left(\phi^o + \frac{1}{2}\,|\boldsymbol{v}|^2 - c_l^2\,\mathrm{d}t\,\nabla\cdot\boldsymbol{v}\right) + \nabla^d \times \left(\boldsymbol{\psi}^o + \frac{1}{2}\,|\boldsymbol{v}|^2\,\mathbf{n} - c_t^2\,\mathrm{d}t\,\nabla\times\boldsymbol{v}\right) \tag{57}$$

The two terms of the right-hand side are two Lagrangians associated, respectively, with the conservation of the compression and rotation accelerations. Noether's theorem applied to laws of physics in the form of Lagrangians or Hamiltonians allows us to invoke the invariances of a mechanical system. In particular the pressure energy defined by ϕ^o and the kinetic energy $1/2\,|\boldsymbol{v}|^2$ can change in the course of time while keeping the total energy. In particular, the total kinetic energy can increase, while the pressure decreases.

It should be recalled that discrete model (9) has allowed to strictly recover the results of the Navier–Stokes equation for fluid flows and those of the Navier-Lamé equation for solids and their coupling for fluid–structure interaction. Turbulence is addressed for the first time with this physical model. The possible differences are analyzed in depth from the properties of the Navier–Stokes equation and the discrete equation of motion.

4.4 Turbulence in Taylor–Green vortex

The case of the Taylor–Green vortex defined in many publications in three space dimensions is different in that the two components of the velocity (u, v) are functions of the z direction. The velocity field $\boldsymbol{v} = u(x, y, z)\,\mathbf{e}_x + v(x, y, z)\,\mathbf{e}_y + w(x, y, z)\,\mathbf{e}_z$ of initial 3D Taylor–Green vortex in the domain $L^3 = [-\pi, \pi]^3$ is of the form:

$$\begin{cases} u = -v_0\,\cos x\,\sin y\,\cos z \\ v = \;\; v_0\,\sin x\,\cos y\,\cos z \\ w = 0 \end{cases} \tag{58}$$

where v_0 is a constant chosen here equal to unity.

The equilibrium pressure field deduced from the equation of motion is written as:

$$\phi^o = \phi_0 + \frac{v_0^2}{16}\,(\cos x + \cos y)\,(\cos z + 2) \tag{59}$$

where ϕ_0 is a constant.

The simulations are performed with an initial field potential $\phi^o = 0$. As the physical model is fully compressible, waves propagate in the field with a longitudinal celerity equal to $c_l = 10^2\ \mathrm{m\ s^{-1}}$, but tests with higher celerities (10^6) do not change anything about the flow behavior. We simply observe that they attenuate very quickly to give solution (59) after some oscillations due to the compressible motion.

The solution of this problem has been obtained many times from various numerical methods. The development of spectral and pseudo-spectral methods notably by Orszag [12,20] and used by many other authors [2] is very well adapted to simulate turbulence. The smallest structures of the turbulence are captured with a high accuracy allowing to represent the energy cascade at high Reynolds numbers. Since then, other high-precision methods have been implemented to find the same results; it is for example the case of the discontinuous Galerkin method [9,25,28] which gives excellent results on the TGV case.

The reference case used as a benchmark to specify the properties of the numerical methods in direct simulation corresponds to a Reynolds number of $\mathrm{Re} = 1600$. At this Reynolds number the smallest scale of viscous dissipation is indeed very small and justifies the choice of efficient methods. The results of the different authors are very close, and a synthesis is available [25], in file "spectral-1600-512.gdiag."

For the results obtained from the Navier–Stokes model the temporal evolution of the kinetic energy is calculated on the volume:

$$E_k = \frac{1}{[\Omega]}\int_\Omega \frac{\rho}{2}\,\boldsymbol{v}\cdot\boldsymbol{v}\,\mathrm{d}v \tag{60}$$

where $[\Omega]$ is the domain volume. The kinetic energy dissipation rate is, for an incompressible flow:

$$\varepsilon = 2\,\nu\,\mathcal{E} = -\frac{d\,E_k}{\mathrm{d}t} = 2\,\nu\int_\Omega \frac{\rho}{2}\,\boldsymbol{\omega}\cdot\boldsymbol{\omega}\,\mathrm{d}v \tag{61}$$

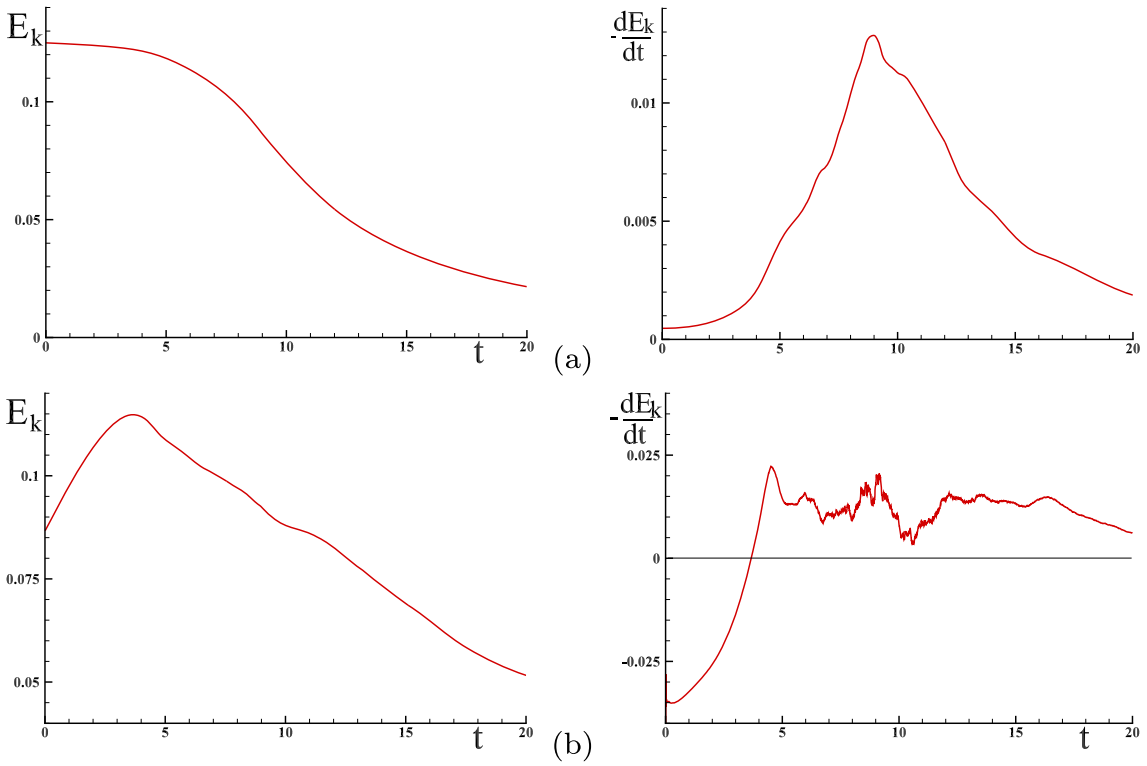

Fig. 9 Taylor–Green vortex at Re = 1600; in top **a** the spectral Re-1600-512.gdiag simulation provides the evolution of dimensionless values of E_k, $\varepsilon = -\mathrm{d}E_k/\mathrm{d}t$ for the Navier–Stokes model [24] and in bottom **b** the same results for the discrete mechanics model

where $\mathcal{E}$ is temporal evolution of the enstrophy integrated on the domain Ω.

For compressible flows other terms must be considered, but as the Mach number is very low, they will be ignored.

For compressible flows other terms must be considered, but will be ignored in the present analysis since it is focused on flows at very low Mach number. The kinetic energy E_k and the dissipation $-\mathrm{d}E_k/\mathrm{d}t$ obtained by a spectral method [25] with a spatial approximation of 512^3 are reproduced in Fig. 9a. We observe a monotonic decay of the kinetic energy from its initial value and an always positive dissipation.

Thus the time derivative of the kinetic energy is written as a function of the dissipation:

$$\frac{\mathrm{d}E_k}{\mathrm{d}t} = -2\,\nu\,\mathcal{E} \tag{62}$$

If the viscosity of the fluid is zero, the global kinetic energy is conserved over time.

A direct simulation has been performed from discrete equation of motion (9) under the same conditions as those defined by many authors for a Reynolds number of Re = 1600 and an initial condition fixed by relations (58). The converged results obtained are given in Fig. 9b. They differ significantly from those obtained by many authors with high-precision numerical methods.

In spite of the different definitions of the calculated turbulent quantities *a posteriori* with those of discrete mechanics (54), the calculation of the velocity and pressure is strictly corresponding and can be compared. There are noticeable differences in these first-order quantities, so there is no need to calculate the higher-order moments. The kinetic energy of the discrete mechanics model increases in the purely inertial region before starting the energy cascade toward the smallest scales. At the same time the magnitude $-\mathrm{d}E_k/\mathrm{d}t$ negative initially becomes positive before becoming globally constant in the course of time to then tend toward zero due to the preponderant viscous effects at this stage.

Before attempting to explain the differences observed between these two models, it is necessary to show that the results of discrete mechanics are not due to numerical artifacts.

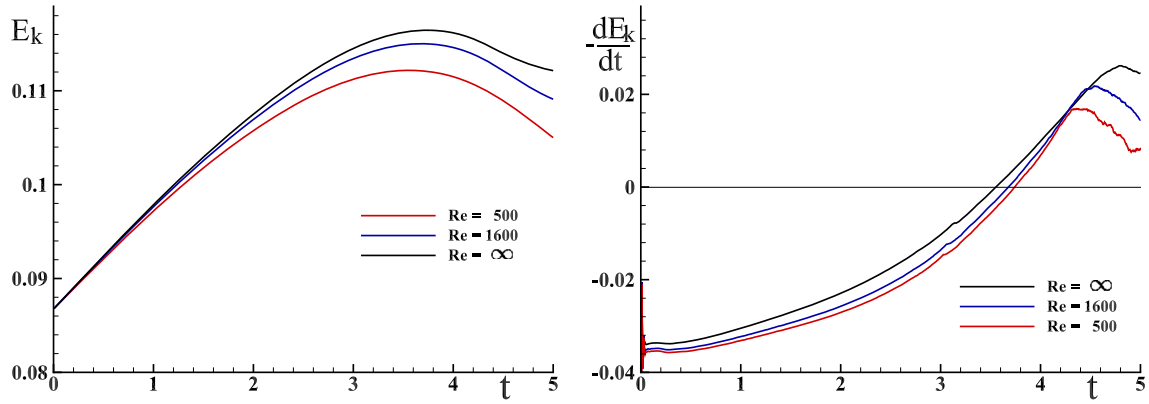

Fig. 10 Re = 500, 1600 and for an inviscid fluid

4.4.1 Convergence in space

In order to confirm the growth of the kinetic energy with time during the inertial phase, simulations were performed at Reynolds numbers of Re = 100, 500, 1600 and for inviscid fluid (10).

A simulation carried out for a Reynolds number of Re = 100 shows that the flow does not have a turbulent character, but the kinetic energy increases, passes through a maximum and decreases monotonically toward zero. On the other hand, the evolutions corresponding to Reynolds numbers of 500 and 1600 are similar; $E_k(t)$ increases, passes through a maximum and then decreases again in the transfer zone. The behavior of the flow for a perfect fluid ($\nu = 0$) is very close to the one corresponding to Re = 1600. On the other hand, beyond an equal time $t = 5$ the kinetic energy increases again and the simulation diverges very quickly; the absence of viscosity does not allow the dissipation necessary to stabilize the flow. Figure 10 on the right shows the evolution of the quantity $-\mathrm{d}E_k/\mathrm{d}t$ for the same values of the Reynolds number; we observe an abrupt break for a time of about $t \approx 4.5$ associated with the point of inflection of the curve $E_k(t)$. Thus the growth of the kinetic energy seems to be a persistent phenomenon independent of the value of the Reynolds number.

The evolutions of the kinetic energy and dissipation for Reynolds numbers of 500 and 1600 being very close, it becomes useless to continue the investigations for the Reynolds number of 1600 which requires higher spatial approximations. Indeed, while the kinetic energy evolutions are perfectly continuous, we observe on Fig. 9 fluctuations on the dissipation evolutions at large time constants which are the sign of an under-resolution. The simulations are performed on a laptop computer with a processor at 2.4 GHz which limits the use of higher approximations. The objective here is to understand the significantly different results of the discrete mechanics compared to the reference model that is the Navier–Stokes equation.

The next step is to remove any doubt on the quality of the results of the simulations performed in discrete mechanics. The numerical method associated with this one is of order two in space and time; the results on many analytical solution cases (Couette flow, Poiseuille flow, etc.), numerical benchmark (steady flow past a circular cylinder, lid-driven cavity, backward facing step, two-phase flows, etc.), show results strictly identical to those of the Navier–Stokes equation [5,8].

Although the numerical methodology is of order two in space and time, many differences exist between the methods. Discrete equation (9) does not require any additional discretization, the differential operators $\nabla \cdot \boldsymbol{v}$, $\nabla \times \boldsymbol{v}$, $\nabla\phi$, $\nabla^d \times \boldsymbol{\psi}$ are used as they are geometrically defined. There is no interpolation, the physical properties; c_l and ν are associated with the operators, respectively, $\nabla \cdot \boldsymbol{v}$ and $\nabla \times \boldsymbol{v}$. The sum of the operators $-\nabla\phi + \nabla^d \times \boldsymbol{\psi}$ is projected on the same segment Γ. The stencil corresponding to an unknown, $\boldsymbol{v}$, associates 74 neighbors that is a linear system where each row has 75 nonzero terms; the number of unknowns is $n_u = 3\,n_e$ where n_e is the number of segments and the number of nonzero terms in the close matrix is $nnc = 75\,n_u$. The nonlinear terms are linearized as $\nabla^d \times (\boldsymbol{v}^n \cdot \boldsymbol{v}^{n+1}/2)$ where $\boldsymbol{v}^{n+1}$ is the solution of the linear system. The solution of this system is realized by a unpreconditioned conjugate gradient method BiCGStab2 in a few tens of iterations whatever the number of degree of freedom. This methodology gives the equation particular properties of robustness and effective accuracy.

A series of simulations are therefore undertaken to verify that the global methodology allows to reach a sufficient accuracy for the Reynolds number of Re = 500. Figure 11 gathers the results as a function of time

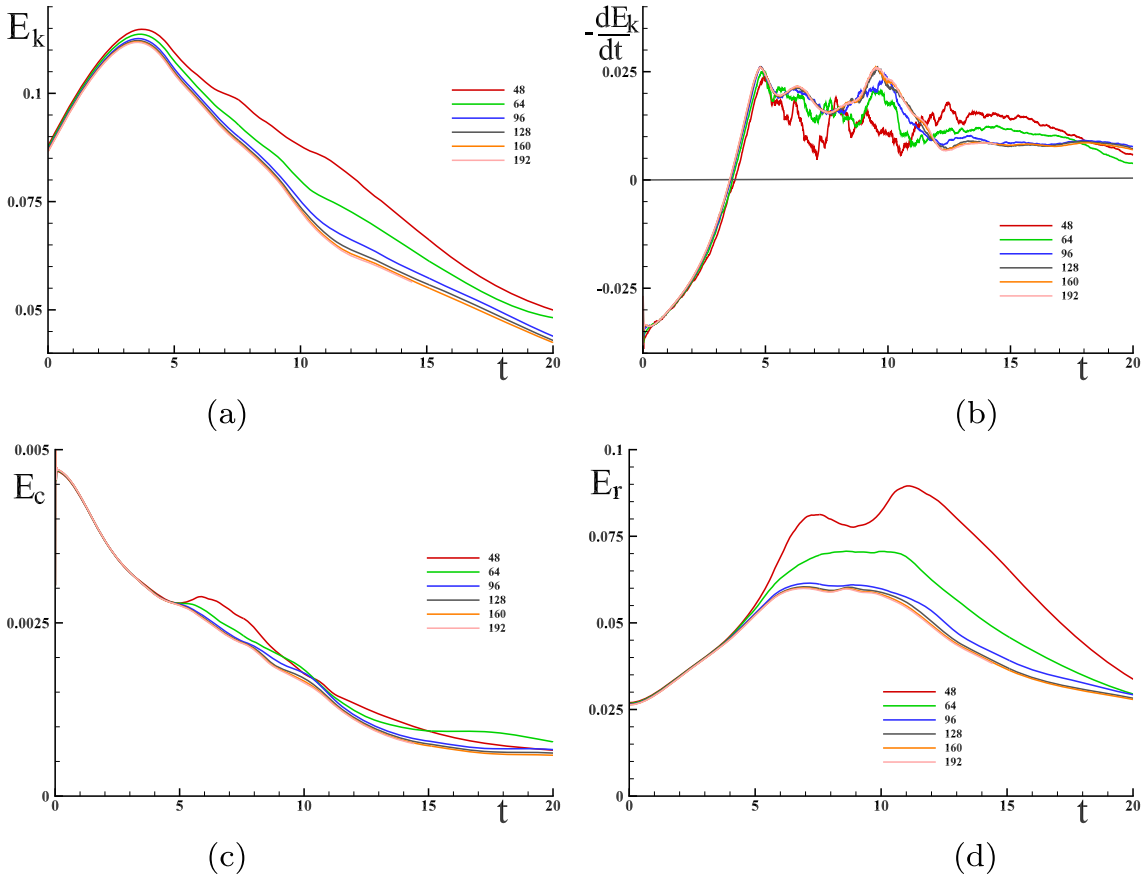

Fig. 11 Re = 500 **a** kinetic energy E_k, **b** $-\mathrm{d}E_k/\mathrm{d}t$, **c** energy of compression E_c and **d** energy of rotation E_r

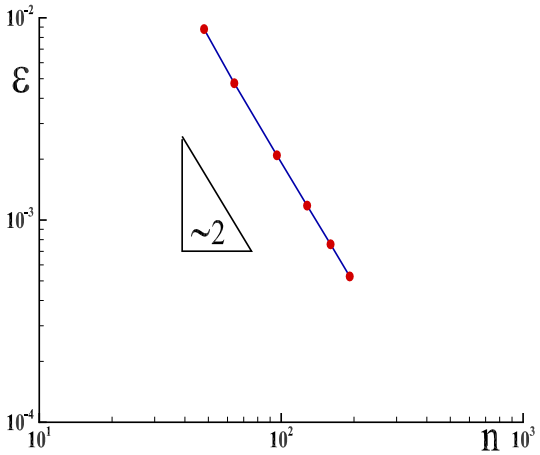

Fig. 12 Convergence in space on the kinetic energy in L_2 norm for Re = 500

on (a) the kinetic energy per unit mass $E_k(t)$, (b) its time variation $-\mathrm{d}E(k)/\mathrm{d}t$, (c) the compression energy E_c and (d) the rotation energy E_r.

All quantities converge as a function of the spatial approximation; it is then possible to calculate an order of convergence. This convergence study is carried out from a Richardson extrapolation on E_k using an error norm L_2. Figure 12 shows indeed a convergence at order two in space on the kinetic energy. These results corresponding to Re = 500 allow to qualify the simulations at least up to this Reynolds number to interpret them from the point of view of the described physics.

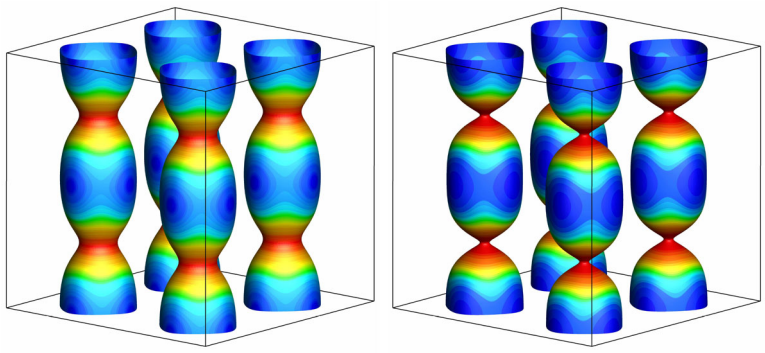

Fig. 13 Vortex-stretching phase for Taylor–Green vortex at Re $= 1600$; snapshots of value of potential $\phi^o = -0.1$ colored by the inertial potential $\phi_i = |\boldsymbol{v}|^2/2$ for $t = 0$ and $t = 0.8$

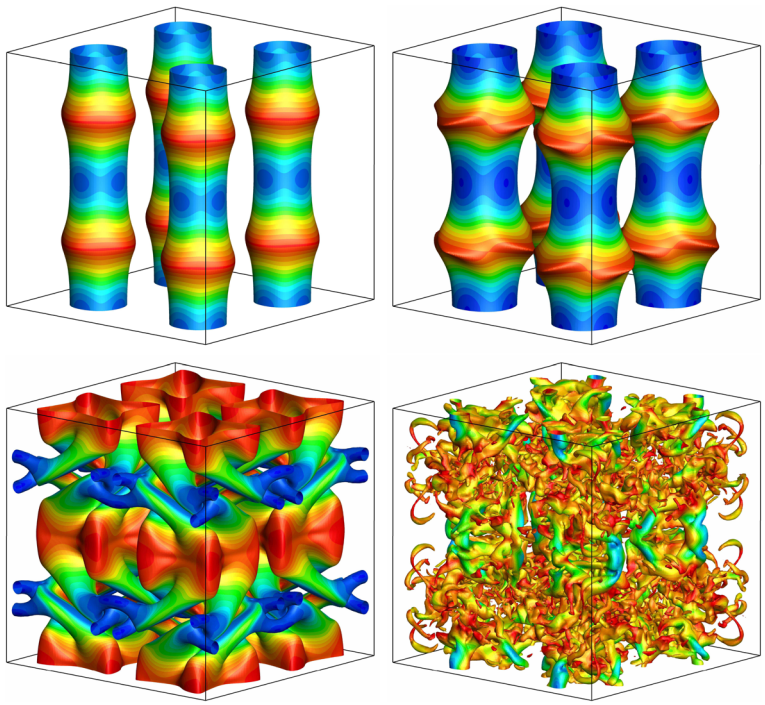

Fig. 14 Taylor–Green vortex for Re $= 1600$ in 196^3; snapshots of iso-values of the ϕ^o_B potential colored by the inertial potential $\phi_i = |\boldsymbol{v}|^2/2$ for $t = 0$, $t = 1$, $t = 5$ and $t = 20$

The vortex stretching phase can be observed for $t \in [0, 3]$ and represented by a pressure value $\phi^o = -0.1$ in Fig. 13 for a Reynolds number of Re $= 1600$. Given the closed geometry of the TGV case, the elongation of the vortices is not possible, but the reduction of the isobaric section with time is very marked. Consequently the rotation velocity, fixed by the scalar ϕ_i, increases significantly which translates into an important increase of the kinetic energy E_k.

The evolution of vortex structures is much better discernible on Bernoulli ϕ^o_B scalar potential fields. Figure 14 shows some snapshots of isovalues of the Bernoulli scalar potential ϕ^o_B superimposed on the inertial potential field $\phi_i = |\boldsymbol{v}|^2/2$ for times $T = 0, 1, 5, 20$ for a Reynolds number of Re $= 1600$. The visual results of the velocity or scalar potential fields are quite comparable to those of the literature on the same subject [10,25].

The initial condition on ϕ^o is set to 0 because this quantity must be subject to the velocity, but after a very short time the solution on ϕ corresponds to the pressure value given by (59). The pressure field for $t = t_b = 1$ reveals the beginning of the vortex bursting process preceded by a vortex-stretching phase is observed where the local velocity increases, the vortex radius reduces and the pressure increases. For a time $t = 3$ the vortex

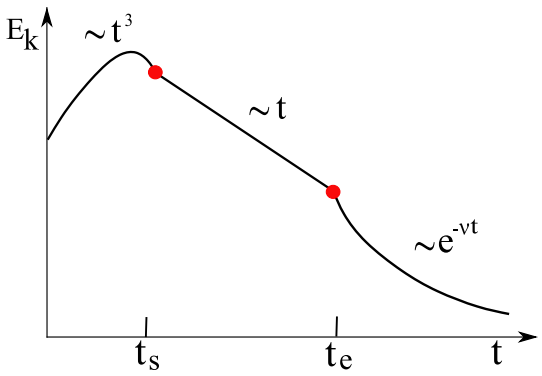

Fig. 15 Scheme of the evolution of the kinetic energy in the inertial phases, the energy cascade and the dissipation zone

bursting phase is where ejections are emitted from the main vortices. The secondary flows are organized into intermingled vortices ($t = 5$) of intermediate sizes. For larger times all structures seem to emit smaller vortices. This scenario is the one described to explain the energy cascade in the transfer phase which is similar to what is described in the literature. Despite the very different kinetic energy evolutions from those observed with the Navier–Stokes model in the inertial zone, the degeneracy of the vortices to smaller scales seems very comparable with previous results.

The analysis of the results of the evolution of the kinetic energy E_k, of $-\mathrm{d}E_k/\mathrm{d}t$ and of the potential ϕ^o as a function of time allows us to find the behavior inscribed within the discrete kinetic energy theorem for inertial phase (47):

$$\frac{1}{2}\frac{\mathrm{d}|\boldsymbol{v}|^2}{\mathrm{d}t} = -\boldsymbol{v}\cdot\nabla\phi^o \tag{63}$$

Indeed the increasing evolution of $-\mathrm{d}E_k/\mathrm{d}t$ is in t^2 and the decreasing one of ϕ^o is also in t^2. The calculation of the time evolution of $\boldsymbol{v}$ shows that the velocity is constant on average. Thus the kinetic energy varies as $E_k \propto t^3$.

Figure 15 traces schematically the variations of the kinetic energy for the inertial phase, the transfer phase and the dissipation phase. From $t = 0$ to $t = t_s$ the inertial phase is accompanied by a dissipation even if it is relatively weak as shown in Fig. 10. The first burst appears for a time $t_b < t_s$, but the transfer phase really starts at time t_s after smaller vortices interact. The evolution of $-\mathrm{d}E_k/\mathrm{d}t$ shows that in the energy cascade this quantity fluctuates but remains almost constant on average, hence satisfying the approximation in t. In the dissipation zone, the quantitative examination of the evolution of the kinetic energy $E_k(t)$ in log-log coordinates shows an exponential law $E_k \propto \exp(-\nu\, t)$.

From the energetic point of view the growth of E_k is closely associated with the decrease of the compression energy E_c; in this phase the angular momentum is indeed conserved. Recall that there is no injection of kinetic energy, the domain is periodic in all directions and no source term is imposed. This energy transfer is essentially inscribed in the entanglement between inertia, compression and rotation effects. In the absence of significant viscous shear the compressional energy is converted to rotational energy, which is what is meant by the vector potential $\boldsymbol{\psi}^o = \nu\,\nabla\times\boldsymbol{v}$ which also carries the block rotation of the vortices. Energy is truly degraded into heat only if the spatial scales are small enough to be dissipated by velocity gradients.

The spectral analysis of the results obtained is for the moment put aside, because it requires simulations at higher Reynolds numbers to significantly recover the undeniable results on turbulence, in particular the $-5/3$ slope of the Kolgomorov theory. The attention is rather focused on the physical analysis of the behavior of the Navier–Stokes equations and the discrete mechanics.

4.4.2 Analysis on the increase of kinetic energy

The quantity $-\mathrm{d}E_k/\mathrm{d}t$ is classically called kinetic energy dissipation rate. In fact, this quantity is not only linked to the dissipation. This classical form of kinetic energy (47) shows that this quantity can vary independently of the viscosity even when $\nu = 0$. In a viscous or nonviscous flow the kinetic energy is not conserved. Equation (9) reflects the preservation of total energy but not of kinetic or potential energy.

No principle justifies a priori that the kinetic energy must decrease monotonically for an inviscid fluid. Let us consider a rotating tube of radius R and length L whose velocity field is given by velocity components (58). The dynamics of this tube is fixed by the conservation of angular momentum expressed in kg m^2 s^{-1} whose transposition in discrete mechanics would be expressed in m^2 s^{-1}; it is the product of the velocity $\boldsymbol{v}$ by the distance between the material point and the rotation axis. This notion of angular momentum $\mathbf{L}_o = \mathbf{OM} \wedge \boldsymbol{v}$ is too restrictive because it is associated with a given velocity. The vector potential $\boldsymbol{\psi} = \nu \nabla \times \boldsymbol{v}$ generalizes this concept by introducing the quantity ν which transforms it into energy per unit mass. Thus when the rotation tube lengthens the potential vector is conserved as its volume and the radius decreases. The velocity $\boldsymbol{v}$ increases as $\boldsymbol{v} \approx 1/R$ and the kinetic energy varies as $E_k \approx 1/R^2$. The mechanical equilibrium between the effects of compression and rotation can only be satisfied if the pressure decreases according to a law $\phi^o \approx t^2$. This instability continues, the rotation velocity increases, the radius R tends to zero and the pressure suddenly increases significantly. The current tube collapses locally and literally explodes producing a vortex bursting. From this moment on, smaller vortexes with an orthogonal axis to the main vortex form in a complex tangle. This phenomenon was described in several publications in particular by Spalart [27] or [18]. More recently the simulation of a single vortex by van Rees in 2020 shows well the mechanism of formation of this phenomenon [24]. More recently, this vortex bursting phenomenon has been observed in vortex filaments of various cross sections initially subjected to localized shear [7]. The overall kinetic energy can increase and decrease with time before dissipating by viscous friction.

It seems that all simulations on the TGV case show a monotonic decrease of the kinetic energy including in the inertial zone. Figure 10 reveals a reduced effect of the viscosity in this zone but does not modify the growth character of E_k with time. This is in discrete mechanics a robust mechanism. The attribution of this essential difference on the evolution of E_k in the inertial zone is thus due to the choice of the physical model, Navier–Stokes or discrete mechanics. It would be risky to draw conclusions about the representativeness of the discrete model for turbulence. It must be noted that both models show exactly the same solutions for classical flows [4,5,8].

However, it is possible to attempt an explanation based on the two equations because they are significantly different. Discrete equation (9) intrinsically ensures the dynamic entanglement between the viscous and rotational effects. This is not the case for the incompressible Navier–Stokes equation because the diffusion of momentum is written as $\nu \nabla^2 \boldsymbol{v} = \nu \nabla(\nabla \cdot \boldsymbol{v}) - \nu \nabla \times \nabla \times \boldsymbol{v}$; as the incompressible model must satisfy the constraint $\nabla \cdot \boldsymbol{v} = 0$ the first term is taken equal to zero without precaution. One must also take into account the term existing in the compressible formulation $\lambda \nabla(\nabla \cdot \boldsymbol{v})$ for which the value of the compression viscosity λ for a fluid is not defined. The Navier–Stokes equation does not intrinsically satisfy the conservation of mass; it must be added to a specific equation $\mathrm{d}\rho/\mathrm{d}t + \rho \nabla \cdot \boldsymbol{v} = 0$ or directly $\nabla \cdot \boldsymbol{v} = 0$. This decoupling inhibits the entanglement phenomenon. The treatment in compressible formulation of the Navier–Stokes equation does not change the result of the monotonic decay of the kinetic energy; it introduces another form of decoupling by integrating an additional state law. Moreover, the majority of simulations performed from the Navier–Stokes equation use a splitting where the pressure is calculated by a separate Poisson equation.

Contrary to classical mechanics where the equations of motion deal with the balance of momentum and momentum, equation (9) contains both notions in a form that translates the conservation of energy per unit mass. The acceleration $\boldsymbol{\gamma}$ carries both the conservation of motion in one direction and that represented classically by the angular momentum of classical mechanics.

4.5 Curvature in space and time of inertial potential

Another explanation can be given for the differences between the continuous and discrete formulations; it concerns the formulation of inertia. In classical mechanics this is written indifferently $\boldsymbol{v} \cdot \nabla \boldsymbol{v}$ or $\nabla \cdot (\boldsymbol{v} \otimes \boldsymbol{v}) - \boldsymbol{v} \nabla \cdot \boldsymbol{v}$ or $\nabla(|\boldsymbol{v}|^2/2) - \boldsymbol{v} \times \nabla \times \boldsymbol{v}$. The application of the differential operators, namely divergence and curl, brings out terms whose meaning is uncertain [8]. In particular, the divergence of the Lamb vector $\boldsymbol{\mathcal{L}} = -\boldsymbol{v} \times \nabla \times \boldsymbol{v}$ produces a term which is the second invariant of the tensor $\nabla \boldsymbol{v}$ which appears as a compatibility condition for solving an incompressible flow. The inertia is written in mechanics in the generic form of a Helmholtz–Hodge decomposition; its divergence and its curl eliminate from the start the dual rotational and the gradient, respectively.

In a certain number of simple flows (Couette, Poiseuille, etc.) the inertial effects are apparently absent. Indeed, for example for the Poiseuille flow the term $\boldsymbol{v} \cdot \nabla \boldsymbol{v}$ is identically zero. But the term $\nabla(|\boldsymbol{v}|^2/2)$ is not, and it is therefore necessary that the second component of the inertia is equal and opposite to the first. This

is the case in continuum mechanics where the Lamb vector is written as $\mathcal{L} = -\boldsymbol{v} \times \nabla \times \boldsymbol{v}$. As the classical formulation is linked to the existence of a global Galilean reference frame, the equality of the absolute values of the two terms imposes that the Lamb vector is also a gradient. This is not the case in discrete mechanics where the gradient of a function can be equal or opposite to the dual curl of a different function. The inertia term is therefore also zero by compensation in discrete mechanics. The material derivative is then written in the form:

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\partial \boldsymbol{v}}{\partial t} + \nabla \left(\frac{1}{2} \, |\boldsymbol{v}|^2 \right) - \nabla^d \times \left(\frac{1}{2} \, |\boldsymbol{v}|^2 \, \mathbf{n} \right) \tag{64}$$

The properties $\nabla \times \nabla \phi = 0$ and $\nabla \cdot \boldsymbol{\psi} = 0$, which are strictly verified whatever the primal and dual structures, allow to filter the equation of motion if necessary in the framework of a decoupling cancelling of course the dynamical entanglement.

The sum of the two components of the inertia noted $\boldsymbol{\kappa}_i = \nabla \phi_i - \nabla^d \times \boldsymbol{\psi}_i$ can be interpreted as the average curvature of the inertial potential $\phi_i = |\boldsymbol{v}|^2/2$ [4]. The analogy between the calculation of the mean curvature as the sum of the two local curvatures of a surface in two orthogonal directions tangent to it seems indeed relevant. In discrete mechanics all physical effects are associated with the Helmholtz–Hodge decomposition in the form of two components, one curl-free and the other divergence-free. The curvature in time of the kinetic energy E_k visible in Fig. 10 for $\nu = 0$ adds a complementary notion. The increase of E_k with time, approximately in $\propto t^3$, takes place with a strong negative curvature. Furthermore, the gravitational potential per unit mass is written $\phi_g = -\mathcal{G} \, M/r$ where $\mathcal{G}$ is the gravitational constant, M is the mass of the body, and r is the distance of the point considered from the center of mass. Its curvature is equal to $\boldsymbol{\kappa}_g = -\nabla \phi_g + \nabla^d \times \phi_g \, \mathbf{n}$. By analogy, if $|\boldsymbol{v}|^2/2$ is the inertial potential, $\boldsymbol{\kappa}_i$ is its curvature in space and $\mathrm{d}E_k/\mathrm{d}t$ its representation in time.

5 Conclusions

J.C. Maxwell's intuition on the dynamic coupling of electric and magnetic fields extends to the modeling of fluid flows or that of solids. The dynamic entanglement of the direct field produced by a pressure difference and that induced field associated with the curl velocity is similar to that described for electromagnetic fields. The equation of discrete mechanics represents both the motions in fluids and solids and the evolution of currents in electromagnetic circuits. The deep nature of the mechanisms of transfer of compressional and shear energies is strictly the same. This is of course the properties of the media that change, the longitudinal and transverse velocities of the material medium or vacuum and the attenuation factors of the waves. These properties are intrinsic to the medium, and it is enough to know them.

In spite of the profound modifications associated with the discrete model, the analytical solutions and the numerical simulations based on the discrete equation are so far exactly the same as those obtained with the Navier–Stokes and Navier–Lamé equation. The discrete formalism reveals in a clear way the phenomenon of dynamical entanglement intrinsically carried by the Helmholtz–Hodge decomposition. However, it is not excluded that differences are revealed during very detailed studies of complex physical phenomena. This could be the case of turbulence in fluids where there are phenomena not completely elucidated.

It is remarkable that indisputable observations do not always lead to the same conclusions or the same models. This is the case of the analogous effects of inertia and gravitation which have led to a principle of equivalence where mass plays an abusive role. Discrete mechanics introduces new concepts based on differential geometry while abandoning some other aspects of classical mechanics. But in no way is it in contradiction with physical observations or general principles.

Declarations

Conflict of interest There is no conflict of interest in this work.

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