Lid-Driven Cavity Flow

In this work the Navier-Stokes equations, written in a ψ(stream-function)-ω(vorticity) formulation, are solved on a two dimensional Cartesian grid, in order to simulate the fluid flow in a square cavity with the upper wall moving at a constant velocity.

Equations

Indicating as $\bar{V}$ the velocity field (being u and v its components in x and y directions, respectively), we have:

$$\vec{\omega} = rot\vec{V} = \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\vec{k} \quad (1) \quad \text{(Vorticity definition)}$$

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x} \quad (2) \quad \text{(Riemann equations)}$$

Combining eq. (1) and (2), we obtain:

$$\omega = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right) = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi$$

$$\boxed{\omega = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}} \qquad (3) \qquad \text{(Poisson equation for the stream function)}$$

We can now find the second necessary equation for solving the initial problem. To this aim, we start from the Navier Stokes equations:

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial P}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2}$$

By taking the curl of the above equation (and after some algebra) we have:

$$\boxed{\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right)} \qquad (4) \qquad \text{(vorticity-transport equation)}$$

The equations we obtained, (1)-(4), can be numerically solved to compute the solution of the stated problem.
To this aim, we adopt a finite difference method on a Cartesian 2-Dimensional grid.
In this way, the discretized form of eq. (1)-(4) is:

- Velocity components:

$$u_{i,j} \cong -\left(\frac{\psi_{i,j+1} - \psi_{i,j-1}}{2\Delta y}\right) \quad \text{and} \quad v_{i,j} \cong \left(\frac{\psi_{i+1,j} - \psi_{i-1,j}}{2\Delta x}\right)$$

- Vorticity transport

$$\omega_{i,j}^{n+1} = \omega_{i,j}^{n} + \Delta t\left[\left(-u_{i,j}^{n}\,\frac{\omega_{i+1,j}^{n}-\omega_{i-1,j}^{n}}{2\Delta x} - v_{i,j}^{n}\,\frac{\omega_{i,j+1}^{n}-\omega_{i,j-1}^{n}}{2\Delta y}\right)\right] + \\ + \frac{1}{\mathrm{Re}}\left(\frac{\omega_{i+1,j}^{n}-2\omega_{i,j}^{n}+\omega_{i-1,j}^{n}}{\Delta x^2} + \frac{\omega_{i,j+1}^{n}-2\omega_{i,j}^{n}+\omega_{i,j-1}^{n}}{\Delta y^2}\right)$$

- Poisson equation

$$\omega_{i,j}^{n} = \left(\frac{\psi_{i+1,j}^{n}-2\psi_{i,j}^{n}+\psi_{i-1,j}^{n}}{\Delta x^2} + \frac{\psi_{i,j+1}^{n}-2\psi_{i,j}^{n}+\psi_{i,j-1}^{n}}{\Delta y^2}\right)$$

(To solve this Poisson equation, one has to rewrite it in the form $\psi=f(\omega)$)

Boundary conditions

Equations (3) and (4) are elliptical partial differential equations. This make necessary to define boundary conditions in each side of the domain.
We can use the following relations:

- Vertical side (Left) :

$$\omega_{1,j} = \frac{0.5}{dx^2}\left(8\psi_{2,j} - \psi_{3,j}\right)$$

- Vertical side (Right) :

$$\omega_{nl+1,j} = \frac{0.5}{dx^2}\left(8\psi_{nl,j} - \psi_{nl-1,j}\right)$$

- Horizontal side (bottom) :

$$\omega_{i,1} = \frac{0.5}{dy^2}\left(8\psi_{i,2} - \psi_{i,3}\right)$$

- Horizontal side (top) :

$$\omega_{i,nh+1} = \frac{0.5}{dy^2}\left(8\psi_{i,nh} - \psi_{i,nh-1}\right) + \frac{3U}{dy}$$

(U is the velocity of the top wall)

Simulations and results

I report here a detailed list of the results I obtained for a simulation of the lid-driven cavity flow at Reynold number equal to 5000 on a 100x100 cartesian grid.

Simulations data	
Mesh	100x100
Reynolds number	5000
BC accuracy (for ω)	2° order
Simulation time step (dt)	10^-3
Tempo fittizio di simulazione	282.67seconds (282670 iterations)

Stream Function

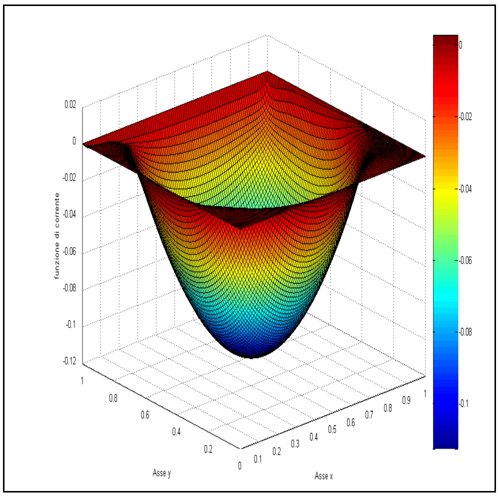


Fig 1: Stream function (3-d surface)
(max=3.5929e-003 ;min=-1.135e-001)

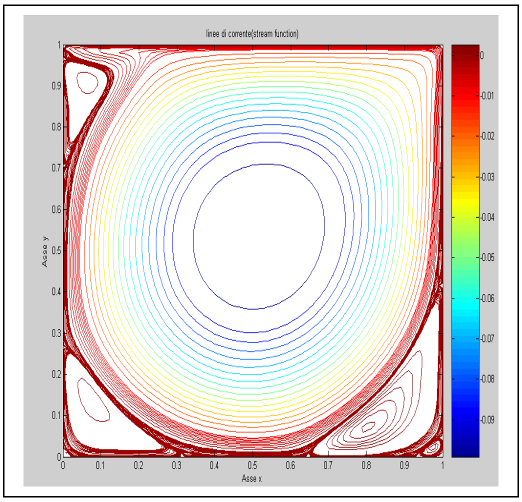


Fig 2: Stream function (isocontours)

Vorticity

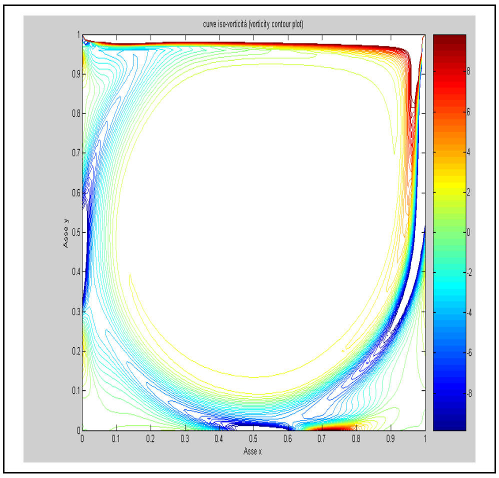


Fig 3: Vorticity (isocontours)

Velocity field

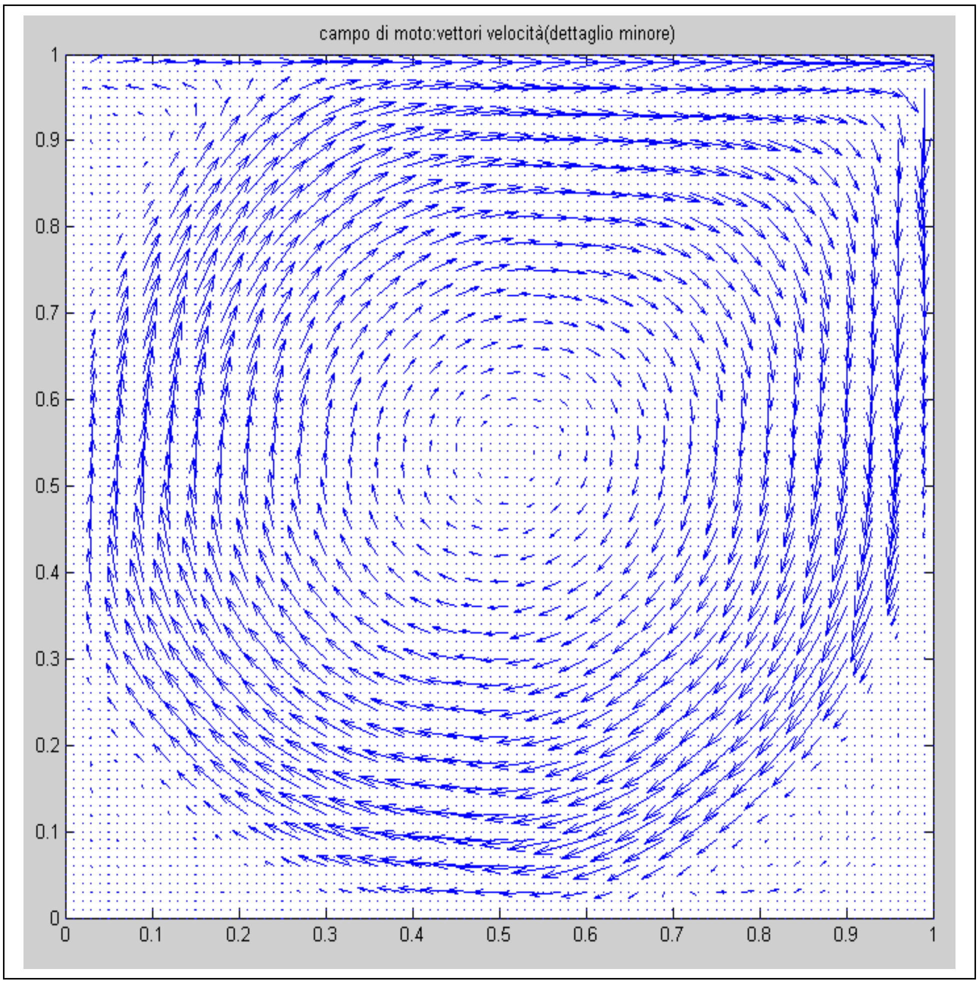


Fig 4: Vector rapresentation of velocity field

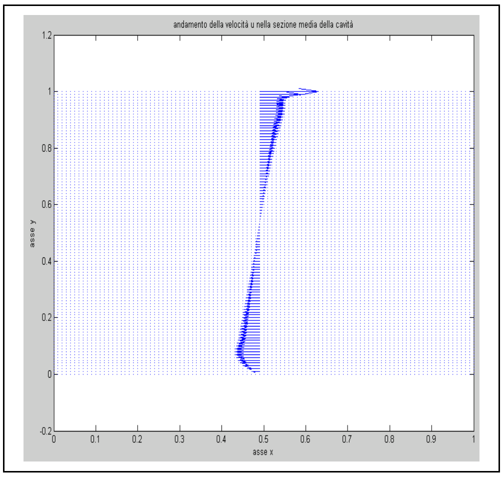


Fig 5: u-component on the middle section

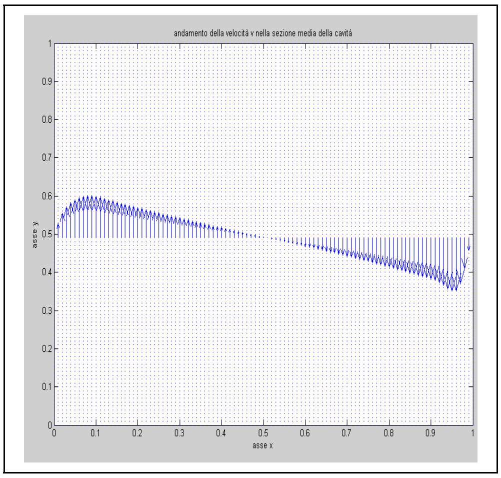


Fig 6:v-component on the middle section

Convergence

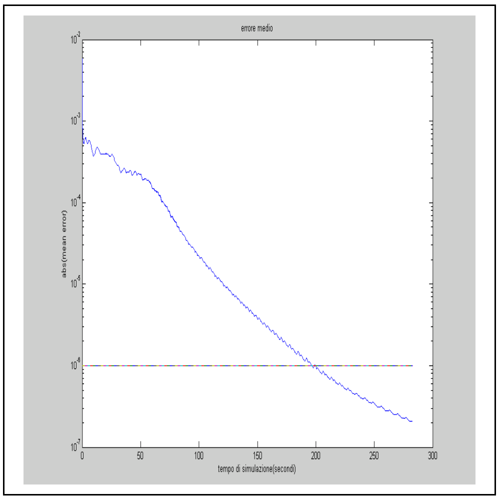


Fig 7:Plot of the "error" as a function of time

In Fig. 7 I plotted the value of the mean difference:

$$\left|\omega^{n+1} - \omega^{n}\right|$$

as a function of the simulation time.

It is necessary to mention that the simulation is stopped once the relation

$$\left|\omega_{i,j}^{n+1} - \omega_{i,j}^{n}\right| < 10^{-6}$$

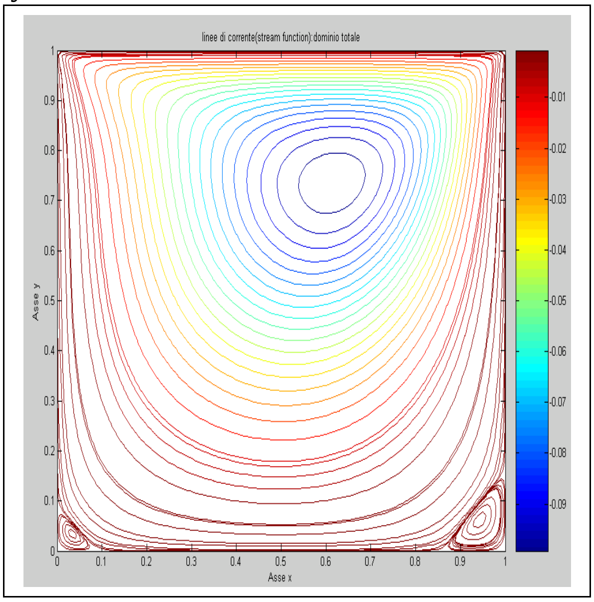
is verified.

In the following section I will present a detailed analysis of the stream function obtained for different simulations (by varying the Reynolds number). These results can be used as a benchmark to compare or validate some new algorithms.

<u>*Streamfunction Analisys*</u>

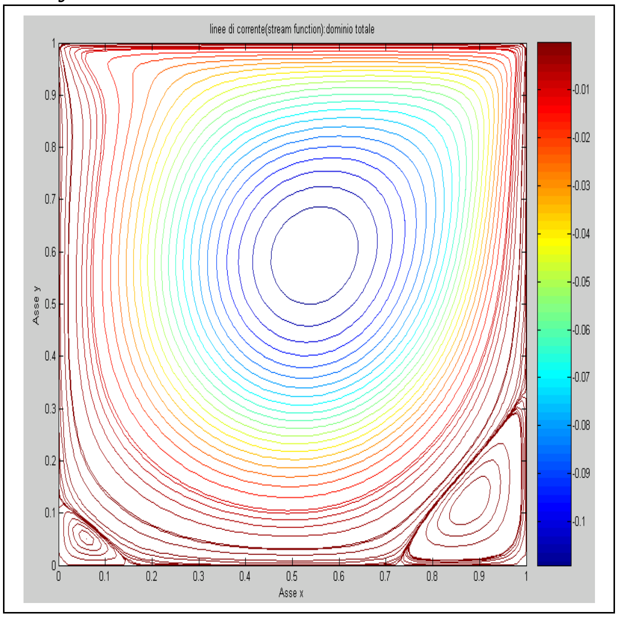
(the tables below refers to the peaks of the stream function)

- First Simulation: Reynolds Number=100



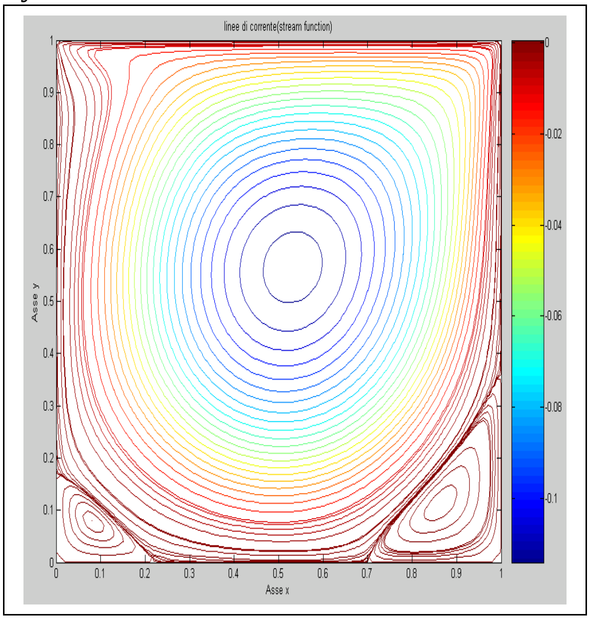
Reynolds	*Primary vortex*	*Upper left vortex*	*Lower left vortex*	*Lower right vortex*	*Secondary lower right vortex*
100	-0.1034	-	$1.7332 \cdot 10^{-6}$	$1.2490 \cdot 10^{-5}$	-

- Second Simulation: Reynolds Number=500



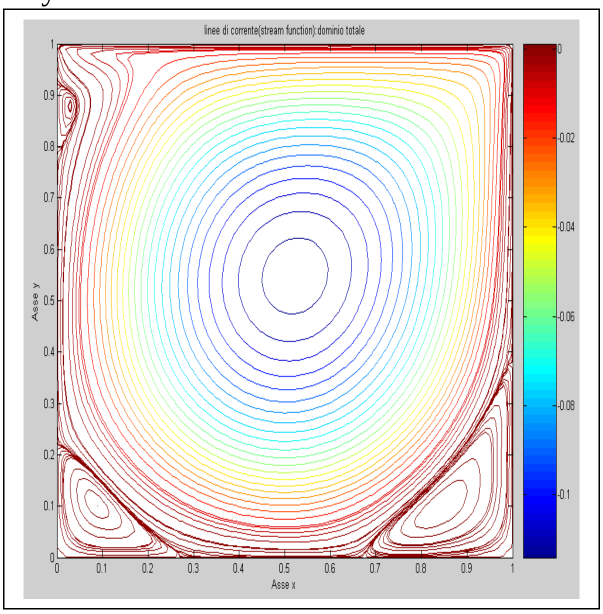
Reynolds	*Primary vortex*	*Upper left vortex*	*Lower left vortex*	*Lower right vortex*	*Secondary lower right vortex*
500	-0.1147	-	$2.8668 \cdot 10^{-5}$	$8.8144 \cdot 10^{-4}$	-

- Third Simulation: Reynolds Number=1000



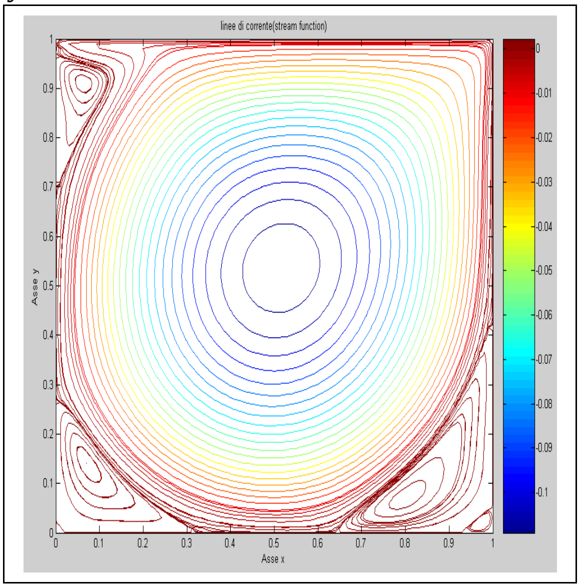
Reynolds	*Primary vortex*	*Upper left vortex*	*Lower left vortex*	*Lower right vortex*	*Secondary lower right vortex*
1000	-0.1173	-	$2.2917 \cdot 10^{-4}$	$1.8 \cdot 10^{-3}$	-

- Fourth Simulation: Reynolds Number=2000



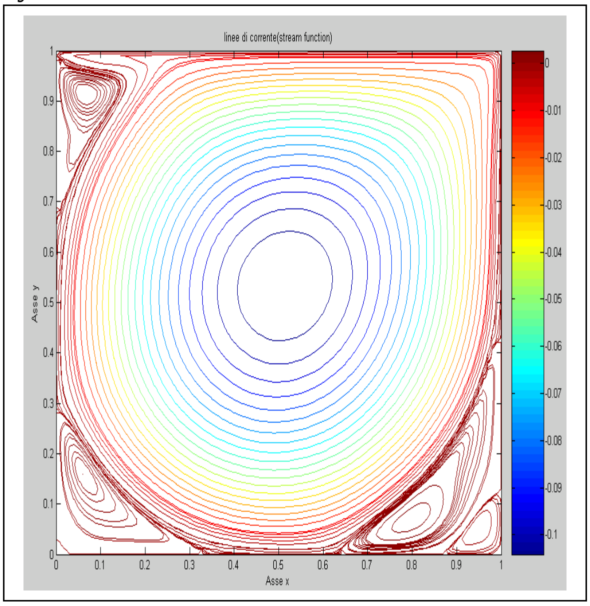
Reynolds	*Primary vortex*	*Upper left vortex*	*Lower left vortex*	*Lower right vortex*	*Secondary lower right vortex*
2000	-0.1175	$1.0134 \cdot 10^{-4}$	$7.3392 \cdot 10^{-4}$	$2.6 \cdot 10^{-3}$	-

- Fifth Simulation: Reynolds Number=5000



Reynolds	*Primary vortex*	*Upper left vortex*	*Lower left vortex*	*Lower right vortex*	*Secondary lower right vortex*
5000	-0.1135	$1.3 \cdot 10^{-3}$	$1.3 \cdot 10^{-3}$	$3.6 \cdot 10^{-3}$	$-3.7995 \cdot 10^{-6}$

- Sixth Simulation: Reynolds Number=7500



Reynolds	*Primary vortex*	*Upper left vortex*	*Lower left vortex*	*Lower right vortex*	*Secondary lower right vortex*
7500	-0.1099	$1.9 \cdot 10^{-3}$	$1.5 \cdot 10^{-3}$	$4.2 \cdot 10^{-3}$	$-8.8889 \cdot 10^{-5}$

Summary of the results (stream function peaks)

Reynolds	*Primary vortex*	*Upper left vortex*	*Lower left vortex*	*Lower right vortex*	*Secondary lower right vortex*
100	-0.1034	-	$1.7332 \cdot 10^{-6}$	$1.2490 \cdot 10^{-5}$	-
500	-0.1147	-	$2.8668 \cdot 10^{-5}$	$8.8144 \cdot 10^{-4}$	-
1000	-0.1173	-	$2.2917 \cdot 10^{-4}$	$1.8 \cdot 10^{-3}$	-
2000	-0.1175	$1.0134 \cdot 10^{-4}$	$7.3392 \cdot 10^{-4}$	$2.6 \cdot 10^{-3}$	-
5000	-0.1135	$1.3 \cdot 10^{-3}$	$1.3 \cdot 10^{-3}$	$3.6 \cdot 10^{-3}$	$-3.7995 \cdot 10^{-6}$
7500	-0.1099	$1.9 \cdot 10^{-3}$	$1.5 \cdot 10^{-3}$	$4.2 \cdot 10^{-3}$	$-8.8889 \cdot 10^{-5}$