Boundary Layer

As a fluid moves past an object, the molecules of the fluid near the object are disturbed and move around the object. In fact, the molecules of the fluid right next to the object's surface stick to it. The molecules just above the surface are slowed down in their collisions with the molecules sticking to the surface. These molecules in turn slow down the flow just above them. The farther one moves away from the surface, the fewer the collisions are affected by the object surface. This creates a thin layer of fluid near the surface in which the velocity changes from zero at the surface to the free stream value away from the surface. This layer is usually called "boundary layer" because it occurs on the boundary of the fluid.
Now, I want to focus on some basilar problems in boundary layer theory.

Boundary layer on an instantaneous accelerated plane

The 2-D equations which control the physic of this problem are:

$$\begin{aligned} &\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} \\ &\frac{\partial P}{\partial y} = 0; \frac{\partial P}{\partial x} = 0; \Rightarrow P = \cos t \end{aligned} \quad (1)$$

After some arrangements one can obtain the following form for the boundary layer equations (for further details, see some specific books):

$$f'' + 2\eta f' = 0 \quad (2)$$

Boundary conditions for the function *f* are:

$$\begin{cases} \eta = 0 \Rightarrow f = 1 \\ \eta \to \infty \Rightarrow f = 0 \end{cases} \quad (3)$$

Eq. (2) with boundary conditions (3) has the following analytical solution:

$$f = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta^2} d\eta \quad (4)$$

The numerical description of the problem is therefore controlled by the following equations:

$$\begin{aligned} &f'' + 2\eta f' = 0 \\ &\eta = \frac{y}{2\sqrt{\nu t}} \\ &u = Uf(\eta) \\ &\delta = 2\sqrt{\nu t} \end{aligned} \quad (5)$$

Here I show some results obtained by the numerical integration of eqs. (5):

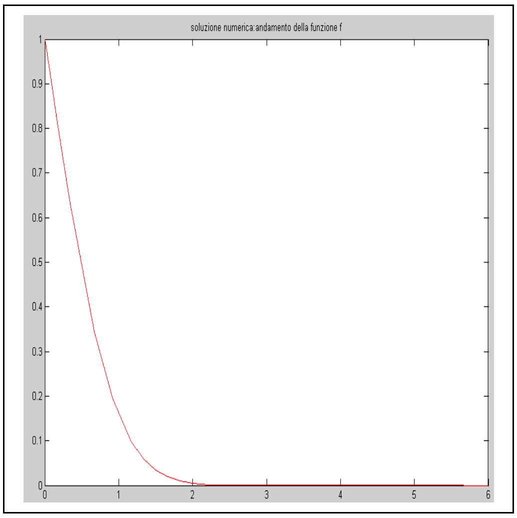

Fig.3: Numerical solution, *f*

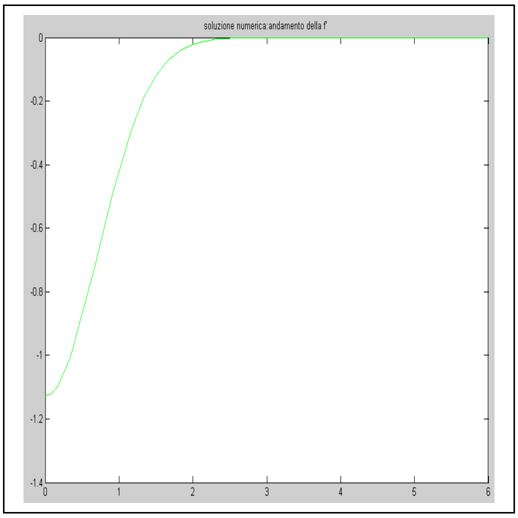

Fig.4: Numerical solution, *f'*

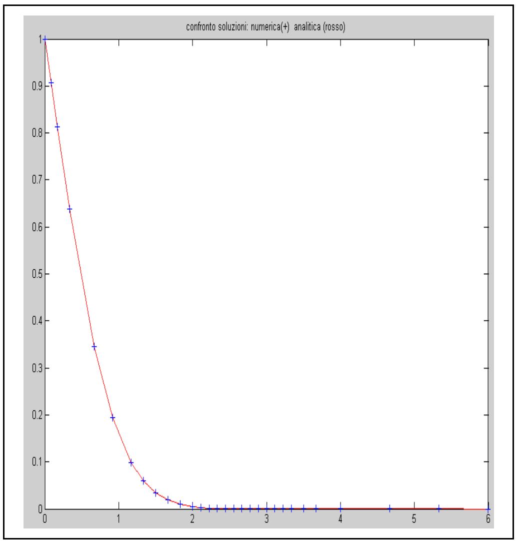

Fig.5: Comparison between numerical (blue dots) and analytical (red) results for function *f*

Analytical	Numerical
1.0000e+000	1.0000e+000
9.0619e-001	9.0618e-001
8.1366e-001	8.1366e-001
6.3735e-001	6.3734e-001
3.4578e-001	3.4577e-001
1.9485e-001	1.9484e-001
9.8960e-002	9.8948e-002
5.9346e-002	5.9339e-002
3.3895e-002	3.3890e-002
1.8422e-002	1.8420e-002
9.5219e-003	9.5207e-003
4.6777e-003	4.6774e-003
2.8306e-003	2.8304e-003
1.6740e-003	1.6739e-003
9.6743e-004	9.6740e-004
5.4629e-004	5.4630e-004
3.0139e-004	3.0141e-004
1.6244e-004	1.6246e-004
8.5523e-005	8.5539e-005
4.3981e-005	4.3993e-005
2.2090e-005	2.2100e-005
1.0836e-005	1.0842e-005
5.1912e-006	5.1951e-006
2.4285e-006	2.4309e-006
7.4310e-007	7.4558e-007
2.1549e-007	2.1691e-007
1.5417e-008	1.8267e-008
4.1209e-011	1.6759e-009
4.6185e-014	1.8103e-010
0	0

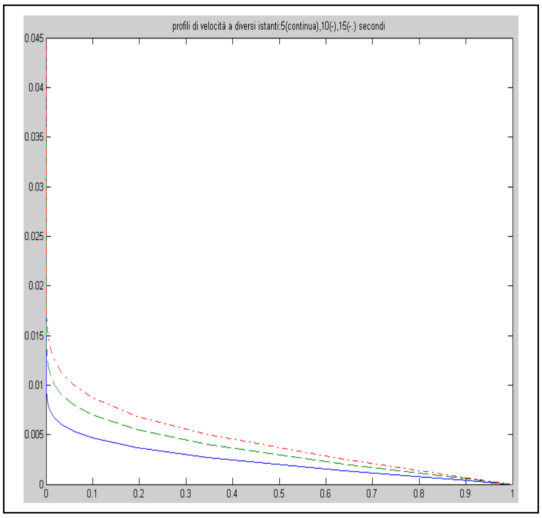

Fig.6: Velocity profiles at different time instants, from blue to red line (5,10,15,sec).

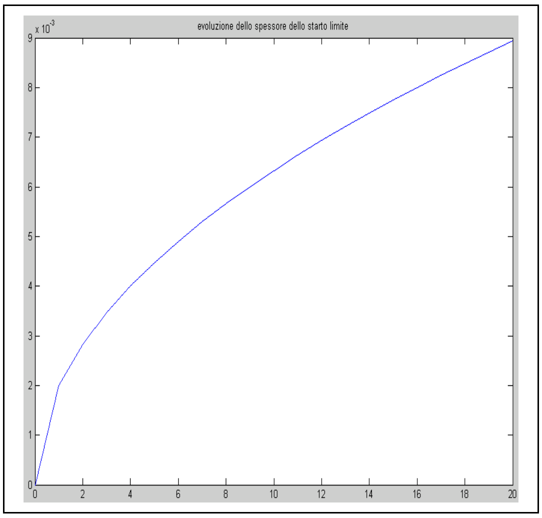

Fig.7: Evolution of boundary layer thickness (δ) in time

Boundary layer over a fixed plane

Now, I want to investigate what happens once a fluid at constant velocity flows over a fixed plane (Blasius boundary layer).
The equations which govern this phenomenon are:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$
$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\mu\frac{\partial^2 u}{\partial y^2} \qquad (1)$$

or, in terms of stream function:

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y}-\frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2}=-\nu\frac{\partial^3 \psi}{\partial y^3} \qquad (2)$$

With the aim of the following variables

$$\eta = y\sqrt{\frac{U}{2x\nu}}$$

$$\psi = \sqrt{2\nu U x}\, f(\eta)$$

$$u = Uf'(\eta) \qquad (3)$$

$$v = \sqrt{\frac{\nu U}{2x}}(\eta f' - f)$$

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_w$$

eq.(2) can be rewritten as:

$$f''' + ff'' = 0 \quad (4)$$

The prescribed boundary conditions are:

$$\begin{cases} \eta = 0 \Rightarrow f = f' = 0 \\ \eta \rightarrow \infty \Rightarrow f' = 1 \end{cases} \quad (5)$$

The results obtained by the numerical integration of eq. (4), with the boundary conditions given by (5), are reported below.

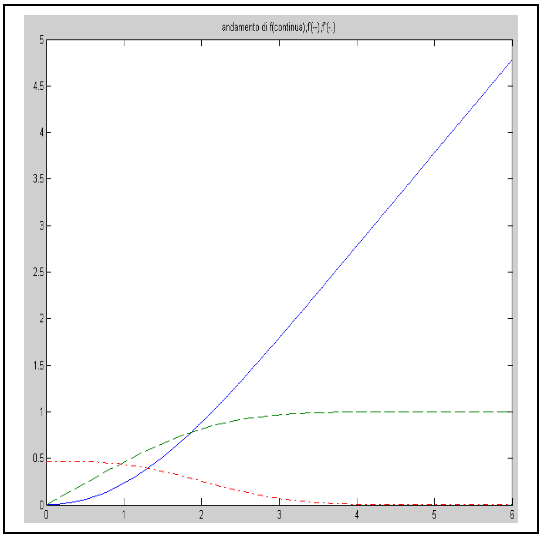

Fig.1: Plot of functions: *f* (blue line), *f'* (green), *f''* (red line)

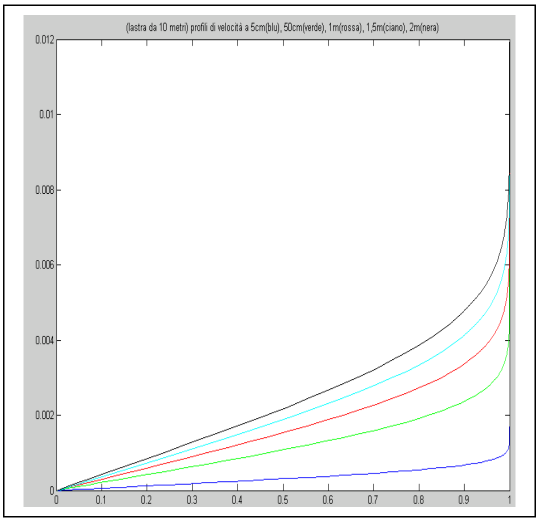

Fig.1: Velocity profiles at different time instants

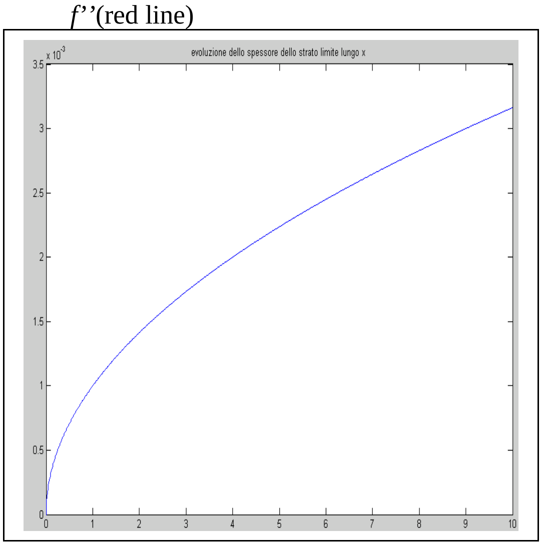

Fig.3: Boundary layer thickness, δ, with respect to axial position

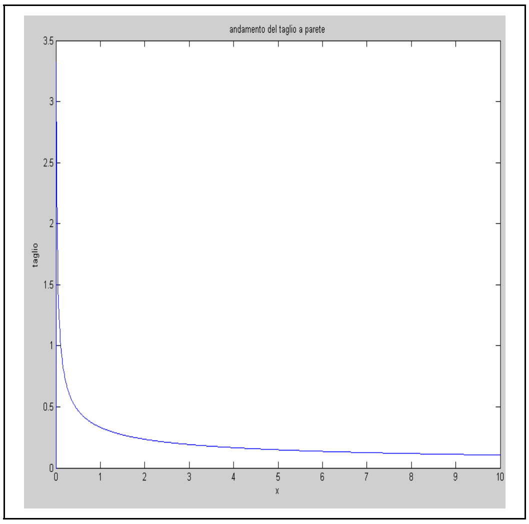

Fig.4: Shear stress at the wall

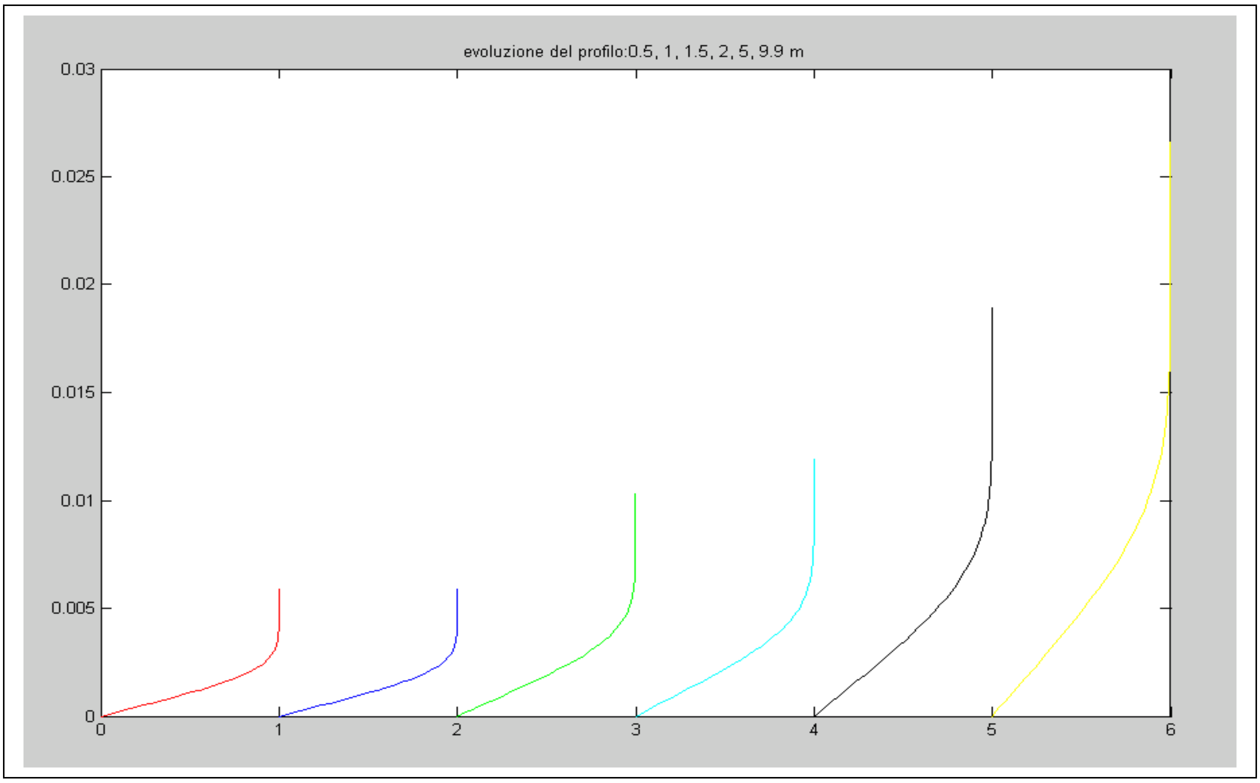

Fig.5: Visualization of velocity profiles at different axial position (0.5, 1, 1.5, 2, 5, 9.9 meters)

Planar jet

Planar jet is another phenomenon that can be treated using the boundary layer theory.

The basilar equations are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

With the usual similarity theory, we can rewrite the above equations as:

$$f''' + ff'' + f'^2 = 0 \quad (1)$$

The results obtained by the numerical solution of eq.(1) with the boundary conditions given by

$$f(0) = f''(0) = 0$$
$$f'(\infty) = 0$$

are presented below.

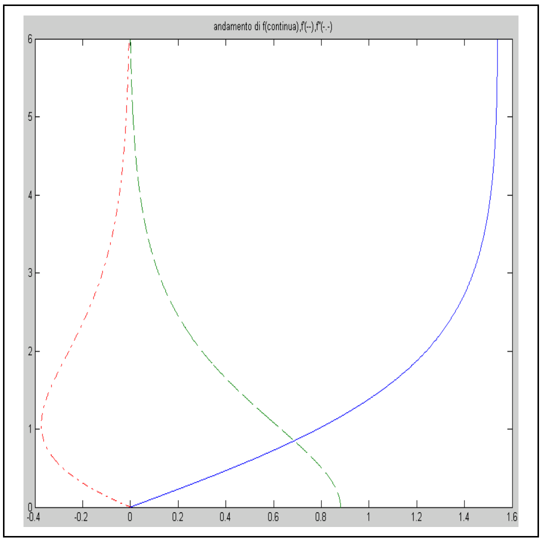

Fig.1: Results for functions *f* (blue), *f'*(green), *f''*(red)

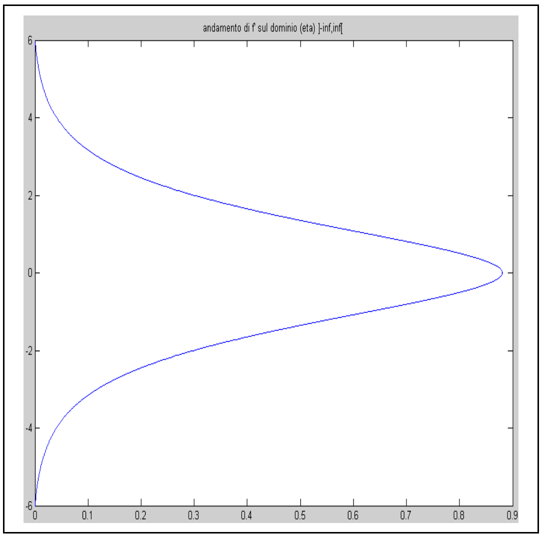

Fig.2: Plot of *f'*

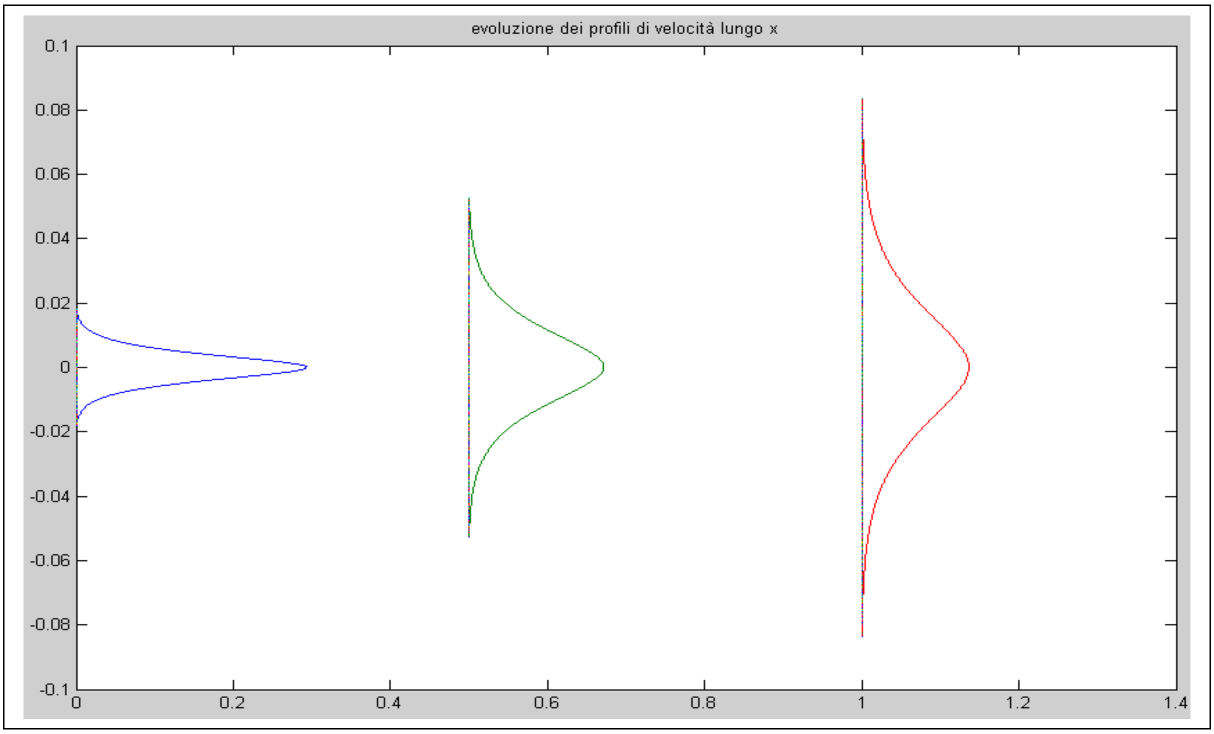

Fig.3: Evolution of velocity profiles with respect of axial position