COST 1005 – Trondheim, October 24th 2012

Effect of particle inertia on the orientation of non-spherical objects in shear flow.

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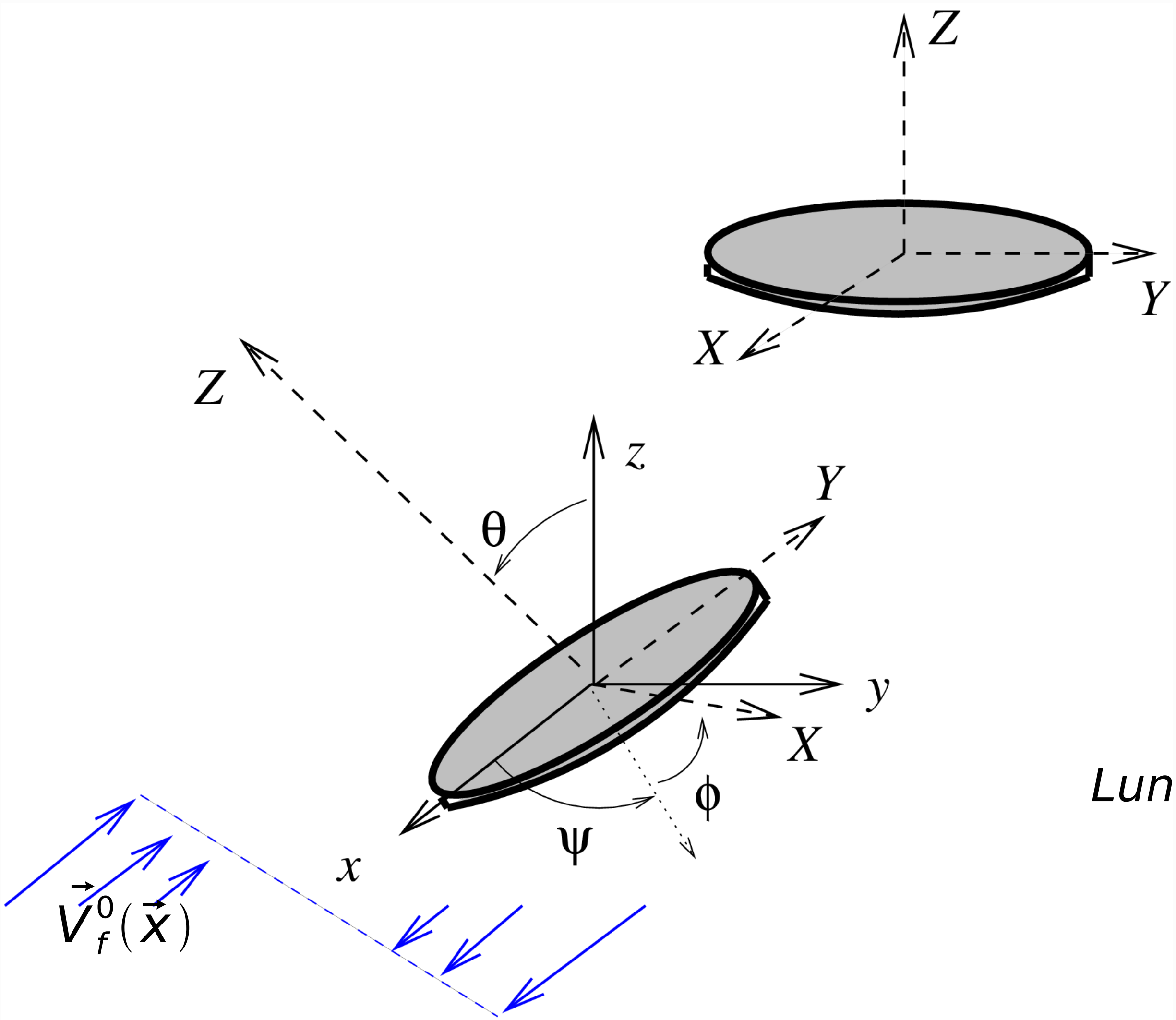


Newtonian fluid

Goal: predict the orientation of non-spherical and non-brownian particles in shear flow.

Analyse the effect of

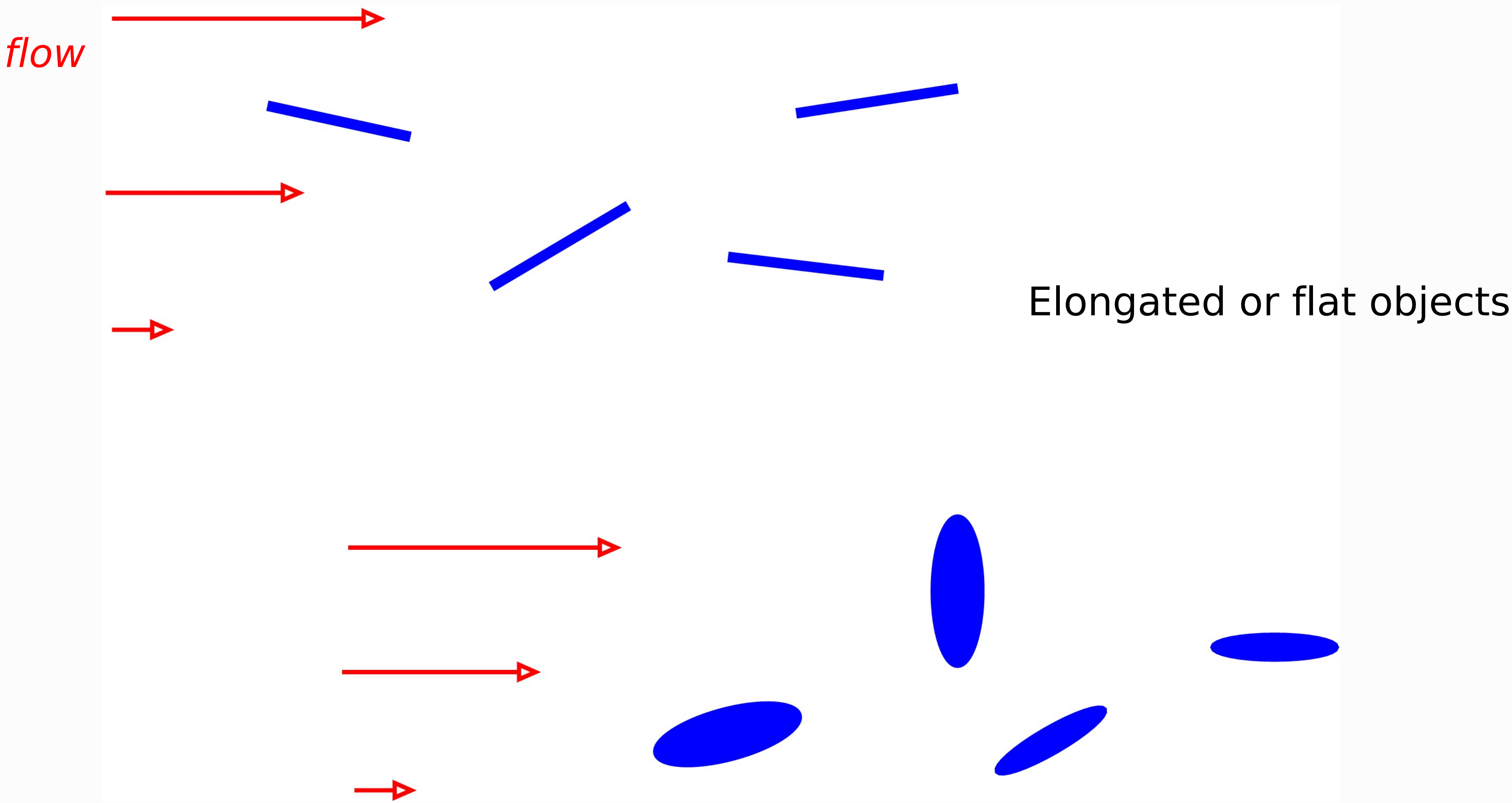
particle inertia :

Broday et al. Phys. Fluids 1998
Subramanian & Koch JFM 2006
Lundell & Carlsson Phys. Rev E. 2010
Lundell Phys. Fluids 2011

Jeffery 1922

Context: physical properties of suspensions

Geosciences: sediments (rock debris, ...)
Engineering: materials, coating systems, papermaking, food, etc.



The orientation of the objects is affected by:

- brownian motion

- hydrodynamic forces

- particle inertia

- electrostatic interactions (walls + neighbouring particles)

- particle/particle interactions (flow modification + collisions)

- ...

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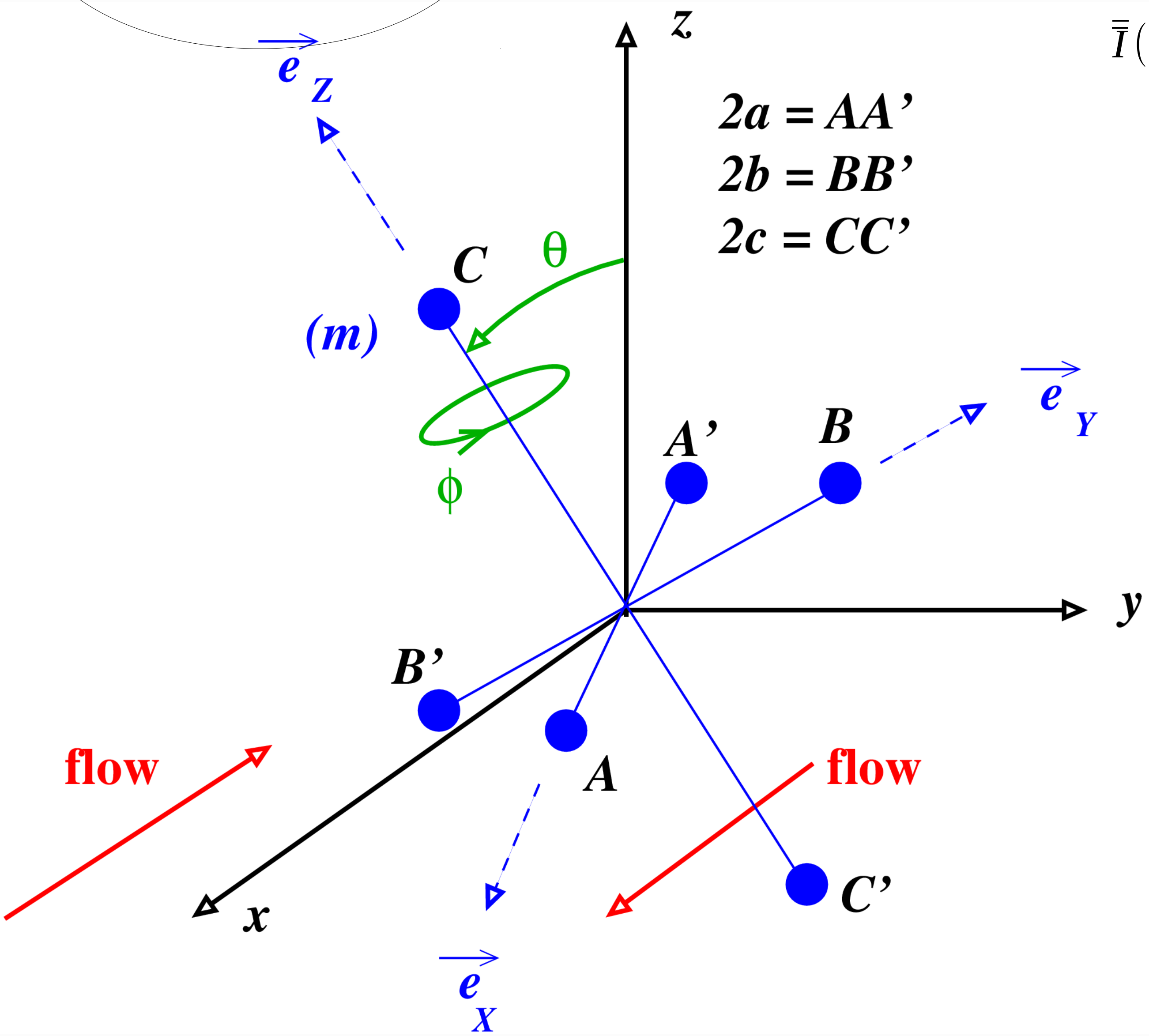
Simple model : triple-dumbbell in a non-uniform flow $\vec{V}_f^0(\vec{x})$

Non-brownian,
low-Re,
isolated
inertial particle



Non-zero inertia tensor:

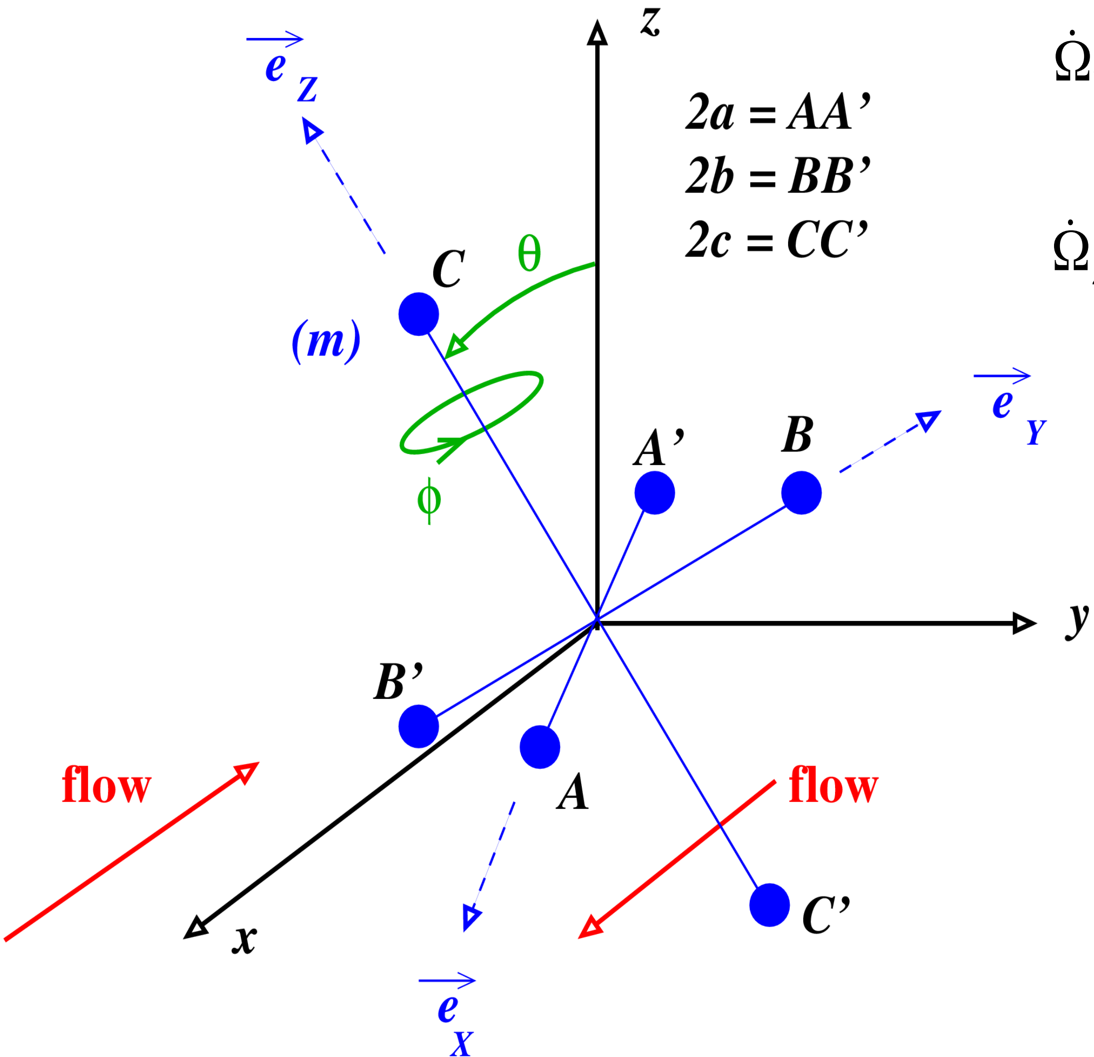
$$\bar{\bar{I}}(O) = 2m \begin{pmatrix} c^2+b^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix}_{XYZ}$$

Suppose each ball experiences a linear drag force:

$$\vec{F}_{fluid \to A} = k \left(\vec{V}_f(A) - \vec{V}_p(A) \right)$$

Simple model : triple-dumbbell in a non-uniform flow $\vec{V}_f^0(\vec{x})$

Small and rigid body:



$$\dot{\Omega}_X + \frac{b^2 - c^2}{b^2 + c^2}\Omega_Y \Omega_Z = \frac{k}{m}\left(\frac{b^2 g_{32} - c^2 g_{23}}{b^2 + c^2} - \Omega_X\right)$$

$$\dot{\Omega}_Y + \frac{c^2 - a^2}{c^2 + a^2}\Omega_X \Omega_Z = \frac{k}{m}\left(\frac{c^2 g_{13} - a^2 g_{31}}{a^2 + c^2} - \Omega_Y\right)$$

$$\dot{\Omega}_Z + \frac{a^2 - b^2}{a^2 + b^2}\Omega_X \Omega_Y = \frac{k}{m}\left(\frac{a^2 g_{21} - b^2 g_{12}}{a^2 + b^2} - \Omega_Z\right)$$

Very close to ellipsoid's equation (*Jeffery 1922).*

where:

$$\vec{\Omega} = \vec{\Omega}_{Body/Labo} = \Omega_X \vec{e}_X + \Omega_Y \vec{e}_Y + \Omega_Z \vec{e}_Z$$

$$g_{12}(\theta, \varphi, \psi) = \vec{e}_X \cdot \overline{\overline{grad}}\, \vec{V}_f^0 \;\; \vec{e}_Y, etc.$$

Simple model : triple-dumbbell in a non-uniform flow $\vec{V}_f^0(\vec{x})$

Non-dimensional form (using a *and* $\dot{\gamma}$):

$$\dot{\Omega}_X + \frac{b^2 - c^2}{b^2 + c^2}\Omega_Y \Omega_Z = \frac{1}{\mathrm{St}}\left(\frac{b^2 g_{32} - c^2 g_{23}}{b^2 + c^2} - \Omega_X\right)$$

$$\dot{\Omega}_Y + \frac{c^2 - 1}{c^2 + 1}\Omega_X \Omega_Z = \frac{1}{\mathrm{St}}\left(\frac{c^2 g_{13} - g_{31}}{1 + c^2} - \Omega_Y\right)$$

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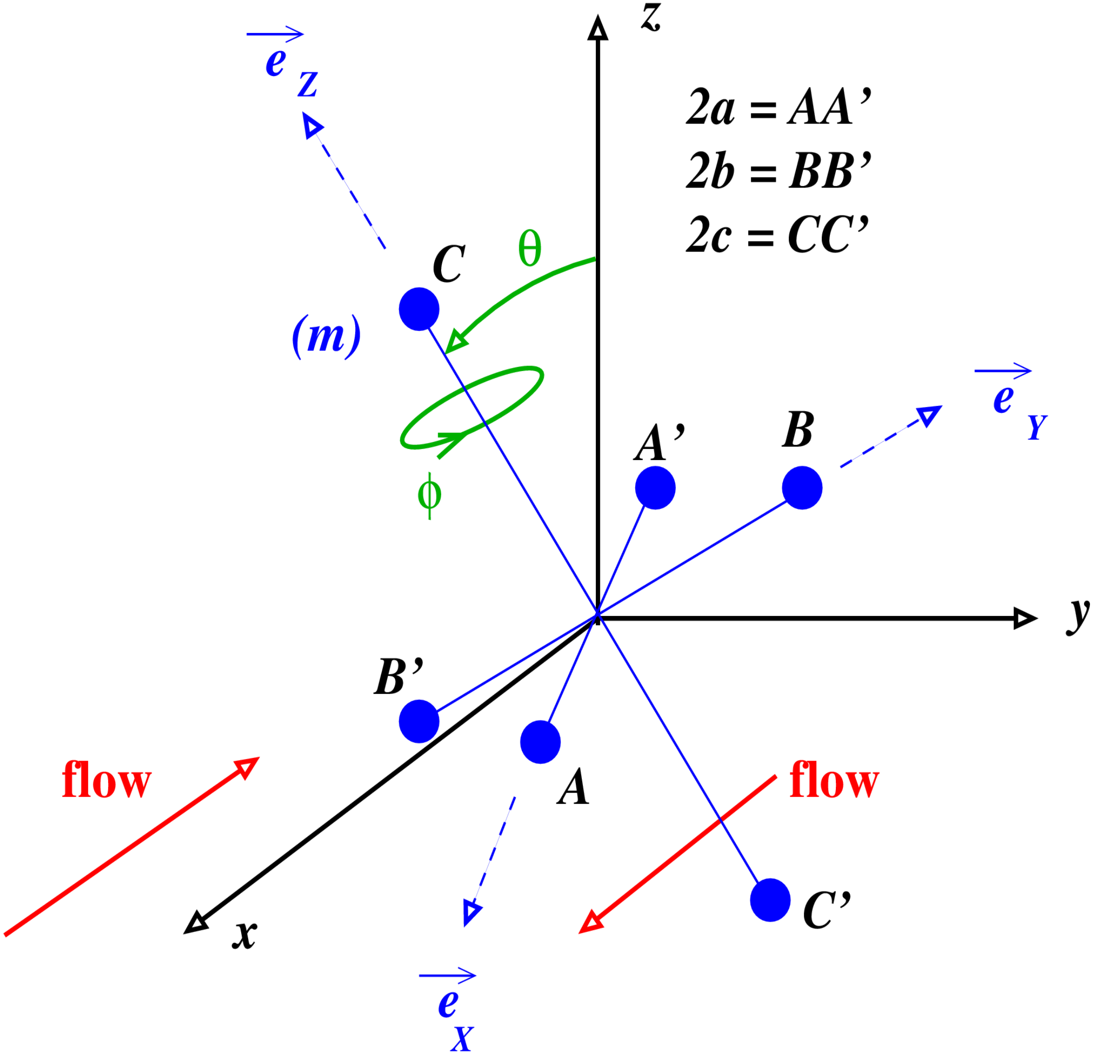

Very close to ellipsoid's equation (*Jeffery 1922).*

Simple model : triple-dumbbell in a non-uniform flow $\vec{V}_f^0(\vec{x})$

Axisymmetric body: $a=b=1$

$$\dot{\Omega}_X + \frac{1-c^2}{1+c^2}\Omega_Y\Omega_Z = \frac{1}{\mathrm{St}}\left(\frac{g_{32}-c^2 g_{23}}{1+c^2} - \Omega_X\right)$$

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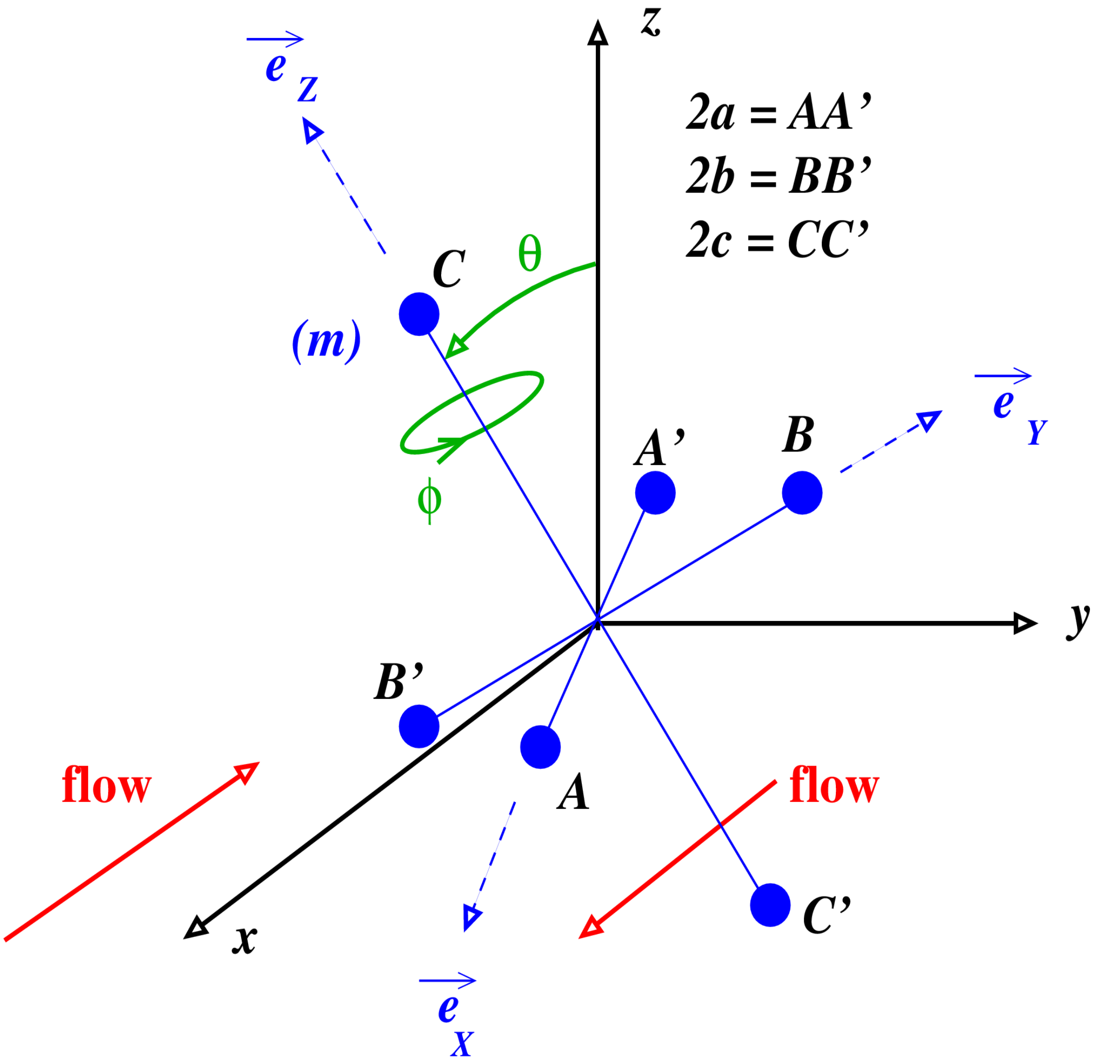

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$$\dot{\Omega}_Z = \frac{1}{\mathrm{St}}\left(\frac{g_{21}-g_{12}}{2}-\Omega_Z\right)$$

6-dimensional phase space:

$$\vec{X}=(\psi,\theta,\varphi)=\textit{(precession,nutation, intrinsic rotation)}$$

$$\vec{Y}=(\Omega_X,\Omega_Y,\Omega_Z)$$

Dynamics of the object:

$$\dot{\vec{X}}=\vec{G}(\vec{X},\vec{Y})$$

$$\dot{\vec{Y}}=\vec{N}_2(\vec{Y})+\frac{1}{\mathrm{St}}\left(\vec{F}_0(\vec{X})-\vec{Y}\right)$$

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6-dimensional phase space:

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Inertia-free limit

Small Stokes number asymptotics St << 1:

Even though Euler angles are very different
from the inertia-free case St = 0,

the component of angular velocity vector
are always close to the inertia-free case:

$$\vec{Y}(t)=\vec{F}_0(\vec{X})+\mathrm{St}\,\vec{F}_1(\vec{X})$$

Leads to an explicit formula for angular velocity components:

$$\vec{Y}(t)=\vec{F}_0(\vec{X})+\mathrm{St}\,\underbrace{\left[\vec{N}_2(\vec{F}_0(\vec{X}))-\nabla\vec{F}_0\,.\,\vec{G}(\vec{X},\vec{F}_0(\vec{X}))\right]}_{\text{inertia effect}}$$

Where $\vec{F}_0$, $\vec{N}_2$, $\vec{G}$ only depend on a, b, c and $\dot{\gamma}$

$\vec{X}$ = *Euler angles of the particle*

$$\vec{Y}=(\Omega_X,\Omega_Y,\Omega_Z)$$

Small Stokes number asymptotics St << 1:

Application to axisymmetric object in shear flow:

$$\vec{Y}(t)=\vec{F}_0(\vec{X})+\mathrm{St}\,\vec{F}_1(\vec{X})$$

takes the form:

precession

$$\dot{\psi}=-\frac{1}{2}\left(1+\frac{c^2-1}{c^2+1}\cos 2\psi\right)$$

$$+\frac{\mathrm{St}}{4}\frac{c^2-1}{(c^2+1)^2}\left[1+(c^2+(c^2-1)\cos 2\psi)\sin^2\theta\right]\sin 2\psi+O(\mathrm{St}^2)$$

nutation

$$\dot{\theta}=\frac{1-c^2}{4(1+c^2)}\sin 2\psi\sin 2\theta$$

$$+\frac{\mathrm{St}}{4}\frac{c^2-1}{(c^2+1)^2}\left[1+c^2+(c^2-1)\cos 2\psi+(1-c^2)\cos 2\psi\cos 2\theta+(1-c^2)\cos 2\theta\right]\sin^2\psi\sin 2\theta+O(\mathrm{St}^2)$$

intrinsic rotation

$$\dot{\varphi}=\frac{1}{2}\frac{c^2-1}{c^2+1}\cos 2\psi\cos\theta$$

$$-\frac{\mathrm{St}}{8}\frac{c^2-1}{(c^2+1)^2}\left[3+c^2+2(c^2-1)\cos 2\psi+(1-c^2)\cos 2\theta+2(1-c^2)\cos 2\theta\cos 2\psi\right]\cos\theta\sin 2\psi+O(\mathrm{St}^2)$$

Small Stokes number asymptotics St << 1:

Without inertia: precession is decoupled

With inertia: precession is affected by nutation

precession

$$\dot{\psi} = -\frac{1}{2}\left(1 + \frac{c^2-1}{c^2+1}\cos 2\psi\right)$$

$$+ \frac{\mathrm{St}}{4}\frac{c^2-1}{(c^2+1)^2}\left[1+(c^2+(c^2-1)\cos 2\psi)\sin^2\theta\right]\sin 2\psi + O(\mathrm{St}^2)$$

nutation

$$\dot{\theta} = \frac{1-c^2}{4(1+c^2)}\sin 2\psi \sin 2\theta$$

$$+ \frac{\mathrm{St}}{4}\frac{c^2-1}{(c^2+1)^2}\left[1+c^2+(c^2-1)\cos 2\psi + (1-c^2)\cos 2\psi \cos 2\theta + (1-c^2)\cos 2\theta\right]\sin^2\psi \sin 2\theta + O(\mathrm{St}^2)$$

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Small Stokes number asymptotics St << 1:

With (small) or without inertia: intrinsic rotation has no effect on precession and nutation

precession

$$\dot{\psi} = -\frac{1}{2}\left(1 + \frac{c^2-1}{c^2+1}\cos 2\psi\right)$$

$$+ \frac{\mathrm{St}}{4}\frac{c^2-1}{(c^2+1)^2}\left[1+\left(c^2+\left(c^2-1\right)\cos 2\psi\right)\sin^2\theta\right]\sin 2\psi + O\left(\mathrm{St}^2\right)$$

nutation

$$\dot{\theta} = \frac{1-c^2}{4\left(1+c^2\right)}\sin 2\psi \sin 2\theta$$

$$+ \frac{\mathrm{St}}{4}\frac{c^2-1}{(c^2+1)^2}\left[1+c^2+\left(c^2-1\right)\cos 2\psi + \left(1-c^2\right)\cos 2\psi \cos 2\theta + \left(1-c^2\right)\cos 2\theta\right]\sin^2\psi \sin 2\theta + O\left(\mathrm{St}^2\right)$$

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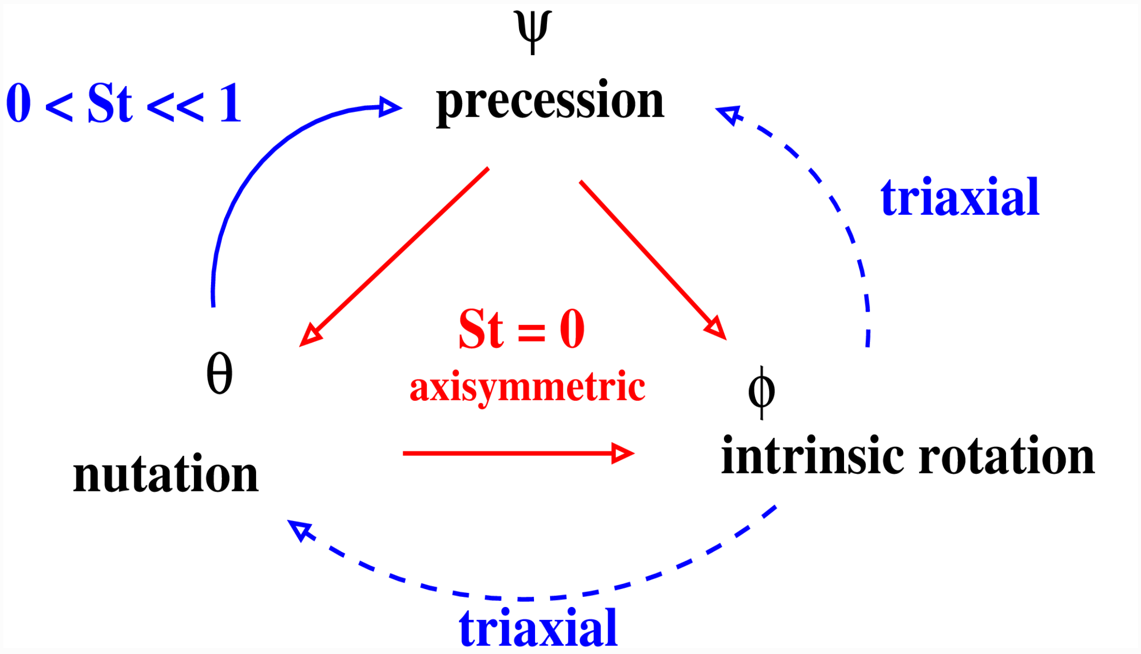
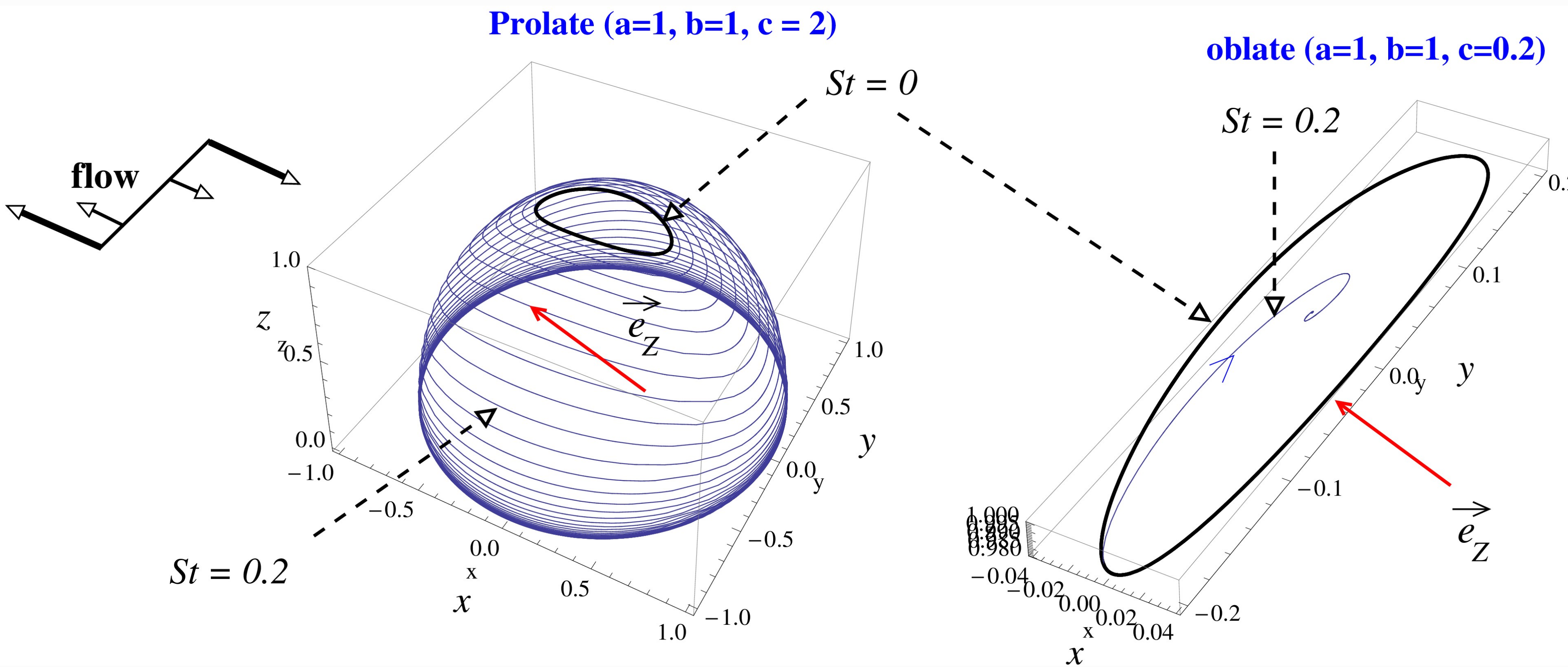
Small Stokes number asymptotics St << 1:



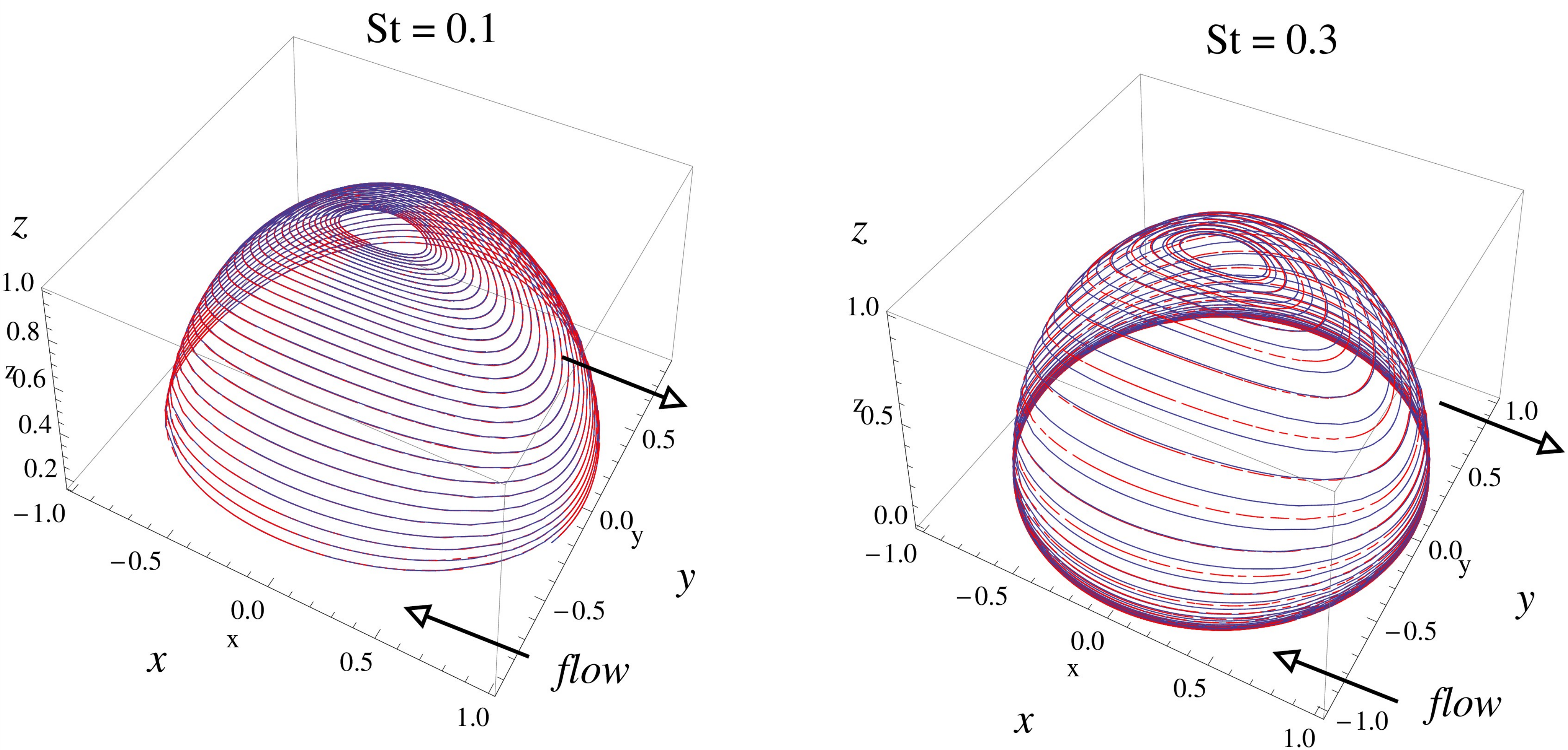
Small Stokes number asymptotics St << 1:

Numerical solution of asymptotic Eq°:



Particle drift towards rotation around small axis *(see Lundell 2011).*

Comparison between numerics and asymptotics (c=2, a=b)



full dynamical equations (6 d.o.f). $(\psi, \theta, \varphi, \Omega_X, \Omega_Y, \Omega_Z)$

asymptotic model (3 d.o.f.). (ψ, θ, φ)

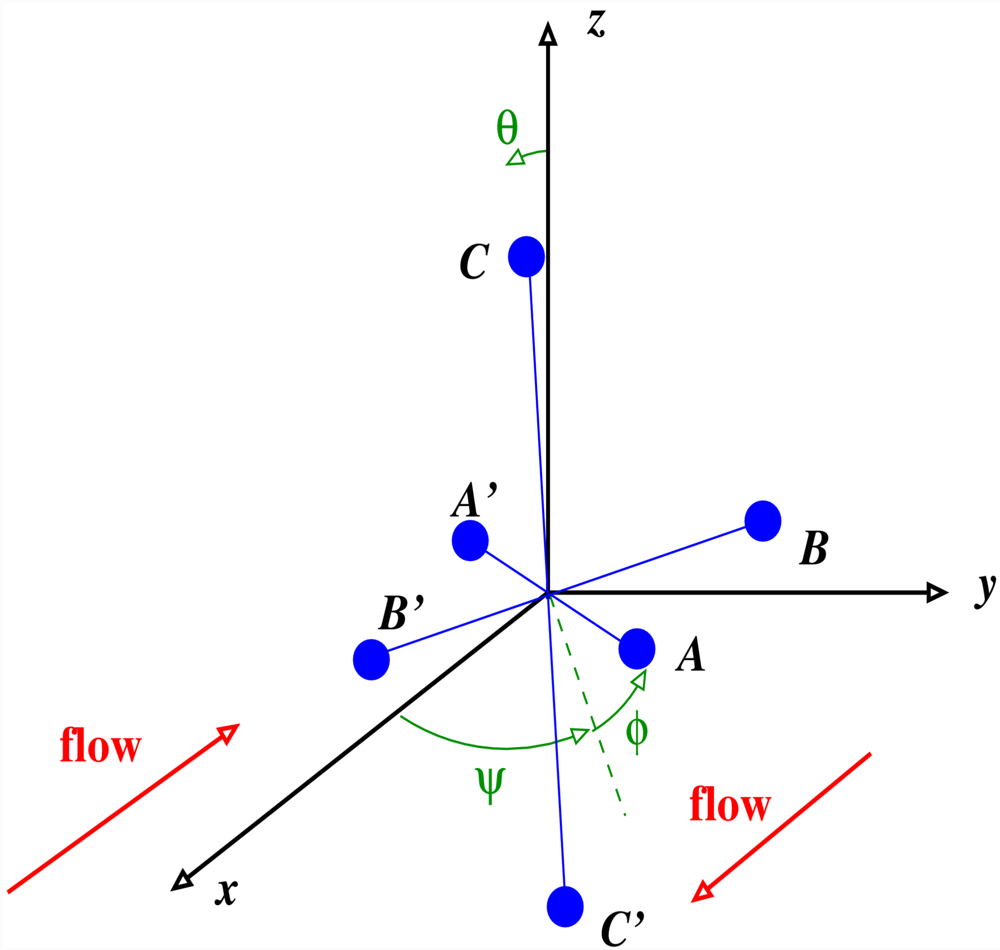
Use asymptotic Eq° for stability analysis:

Linearize particle dynamics near rotation around c axis:

$T =$ period of Jeffery's orbits (St = 0)

$\psi_0(t) =$ T-periodic precession in Jeffery's orbits

Near 'vertical' position: $\theta(t) \approx 0$ and $\psi(t) = \psi_0(t) + \mathrm{St}\,\psi_1(t) + O(St^2)$



$$\ln \frac{\theta(T)}{\theta(0)} = \mathrm{St} \int_0^T [K_0'(\psi_0)\,\psi_1(t) + K_1(\psi_0(t))]\,dt$$

Obtained by solving the O(St) asymptotic Eq°

$$\frac{\theta(T)}{\theta(0)} = \exp\left[\pi\,\mathrm{St}\,\frac{c^2-1}{c^3+c}\right]$$

Use asymptotic Eq° for stability analysis:

→ $$\frac{\theta(nT)}{\theta(0)} = \exp[nT \times \lambda] \;\text{ with }\; \lambda(\mathrm{St}, c) = \frac{\mathrm{St}}{2} \frac{(c^2 - 1)}{(c^2 + 1)^2}$$

→ “vertical” position is stable if c < 1 (flat object), unstable otherwise:

leading to rotation around short axis for long times.

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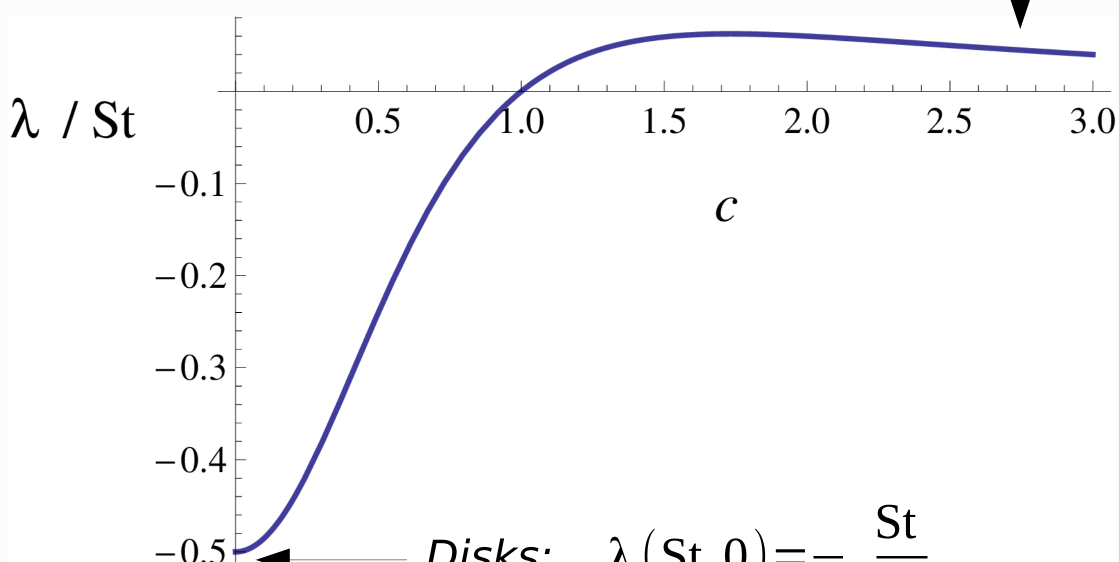


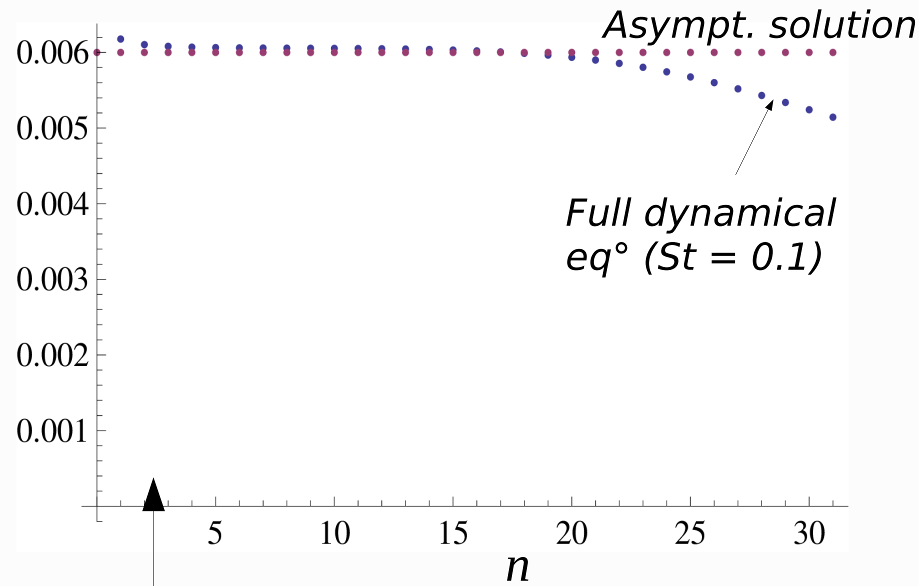
Rods:

$$\lambda(\mathrm{St}, \infty) = 0$$

Disks: $\lambda(\mathrm{St}, 0) = -\dfrac{\mathrm{St}}{2}$

Use asymptotic Eq° for stability analysis:

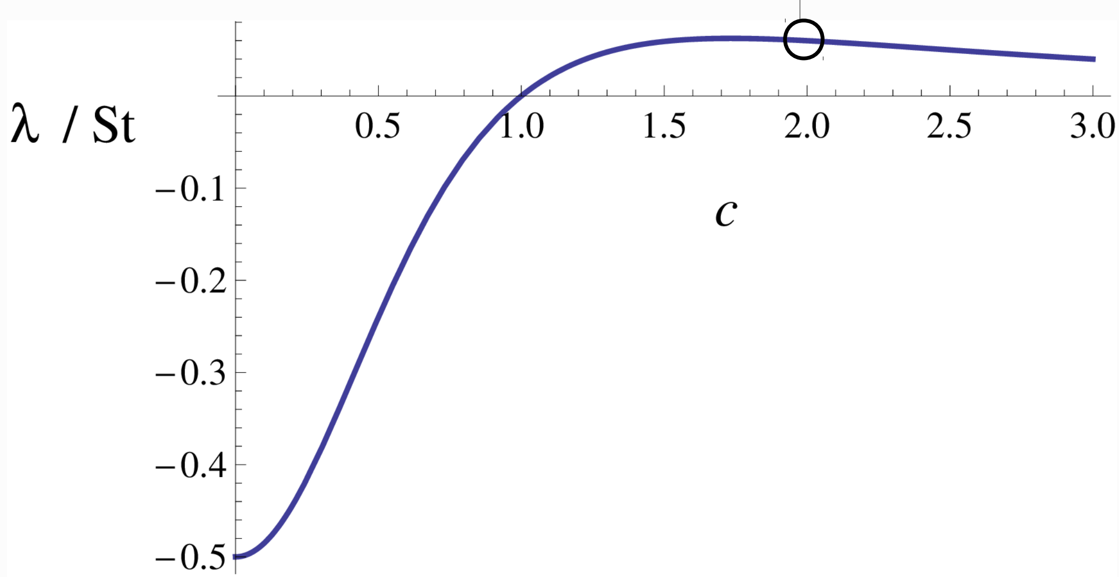
$$\frac{1}{nT}\ln\left(\frac{\theta(nT)}{\theta(0)}\right)$$



$$\frac{\theta(nT)}{\theta(0)}=\exp[nT\times\lambda]$$

Floquet exponent:

$$\lambda(St,c)=\frac{St}{2}\frac{(c^2-1)}{(c^2+1)^2}$$



Conclusion

- An asymptotic expansion of the angular velocity of axisymmetric object has been presented.

- Very cheap way to introduce (weak) inertia effects in fiber suspension computations.

- Theoretical observations:

- precession is coupled with nutation when St > 0

- intrinsic rot° remains decoupled from both precession and nutation (full coupling needs trixiality or larger St).

- enables analytical calculation of Floquet exponent of periodic orbits.

- Perspective: add triaxiality to the asymptotic inertial model.