# On the combined effect of inertia and brownian motion on the orientation of rods in a shear flow. Asymptotic and stochastic analyses.

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# Introduction

- Many flows carry tiny solid objects with a very small particle Reynolds number.
- Geophysical flows : aerosols, sediments, etc.
- Industrial flows : chemical engineering, combustion, etc.
- In many cases particles are non-spherical: rods, platelets, ...
- In addition, particles can have a small but non-negligible inertia.

# Axisymmetric ellipsoid or dumbbell in a simple shear flow

- In the absence of inertia and brownian motion, these particles have periodic orientations (Jeffery 1922).
- **Goals:**
  1. Investigate the stability of the Jeffery orbits when inertia is non-negligible (though small).
  2. Analyze the effect of brownian motion.

*Lundell 2010, Lundell & Carlsson 2011, Subramanian & Koch 2006, Nilsen & Andersson 2013, ..*

# Motion equations

- Axisymmetric ellipsoid with semi-axes $a, a, c$. $\lambda = c/a$.
- Angular velocity vector with respect to the laboratory frame: $\vec{\Omega}(t)$
- Its components in a cartesian basis attached to the object: $\Omega_X, \Omega_Y, \Omega_Z$ (Z : symmetry axis).
- Jeffery 1922 (ellipsoids), set non-dimensional by using $a$ and $\dot{\gamma}$:

$$\dot{\Omega}_X + \frac{1-\lambda^2}{1+\lambda^2}\Omega_Y\Omega_Z = \frac{E_1(\lambda)}{\mathrm{St}}\left(\frac{g_{32}-\lambda^2 g_{23}}{1+\lambda^2} - \Omega_X\right), \quad (1)$$

$$\dot{\Omega}_Y + \frac{\lambda^2-1}{\lambda^2+1}\Omega_X\Omega_Z = \frac{E_2(\lambda)}{\mathrm{St}}\left(\frac{\lambda^2 g_{13}-g_{31}}{1+\lambda^2} - \Omega_Y\right), \quad (2)$$

$$\dot{\Omega}_Z = \frac{E_3(\lambda)}{\mathrm{St}}\left(\frac{g_{21}-g_{12}}{2} - \Omega_Z\right) \quad (3)$$

where $\mathrm{St} = \frac{\dot{\gamma}a^2}{\nu}\frac{\rho_p}{\rho_f}$ and $g_{ij}$ = components of the fluid velocity gradient in the cartesian vector basis attached to the object.

# The $E_i$ coefficients

They appear in front of each component of the torque (Jeffery 1922):

$$E_1(\lambda) = \frac{20}{c(b^2\beta_0 + c^2\gamma_0)}, \tag{4}$$

$$E_2(\lambda) = \frac{20}{c(a^2\alpha_0 + c^2\gamma_0)} = E_1 \quad \text{since } a = b, \tag{5}$$

$$E_3(\lambda) = \frac{20}{c(b^2\beta_0 + a^2\alpha_0)} \tag{6}$$

For dumbbells: $E_i(\lambda) \equiv 1$.

# Asymptotic Analysis when $\mathrm{St} \ll 1$.

- Let $\vec{X} = (\psi, \theta, \phi)$=(precession,nutation,intrinsic rot), and $\mathbf{E}(\lambda) = diagonal(E_1, E_2, E_3)$.
- Orientational motion equations (1)-(2)-(3) now read:

$$\dot{\vec{\Omega}} = \vec{N}_2(\vec{\Omega}) + \frac{1}{\mathrm{St}}\mathbf{E}\left(\vec{F}_0(\vec{X}) - \vec{\Omega}\right) \tag{7}$$

where:

$$\dot{\vec{X}} = \mathbf{L}(\vec{X})\,\vec{\Omega} \tag{8}$$

is the classical relation between angular velocity and derivatives of the Euler angles.

# Asymptotic Analysis when St $\ll$ 1.

- When the Stokes number is very small, $\vec{\Omega} \simeq \vec{F}_0$ (inertia-free limit):

$$\vec{\Omega} = \vec{F}_0(\vec{X}) + \mathrm{St}\,\vec{F}_1(\vec{X}) + O(\mathrm{St}^2). \tag{9}$$

- Leads to an explicit expression for $\vec{\Omega}$

$$\vec{\Omega} = \vec{F}_0(\vec{X}) + \mathrm{St}\ \mathbf{E}^{-1}\left[\vec{N}_2(\vec{F}_0(\vec{X})) - \mathbf{D}\mathbf{L}\,\vec{F}_0(\vec{X})\right] \tag{10}$$

where $\mathbf{D} = \partial\vec{F}_0/\partial\vec{X}$ is the Jacobian of $\vec{F}_0$.

- and for $\dot{\vec{X}} = (\dot{\psi}, \dot{\theta}, \dot{\phi})$

$$\dot{\vec{X}} = \mathbf{L}\,\vec{F}_0(\vec{X}) + \mathrm{St}\,\mathbf{L}\,\mathbf{E}^{-1}\left[\vec{N}_2(\vec{F}_0(\vec{X})) - \mathbf{D}\mathbf{L}\,\vec{F}_0(\vec{X})\right] \tag{11}$$

# Application to the simple shear

- In the case of a simple shear flow of the form $\vec{u}(x, y, z) = \dot{\gamma}\, y\, \hat{\vec{x}}$ in a cartesian frame $(x, y, z)$ attached to the laboratory,
- we obtain the precession and nutation velocities:

$$\dot{\psi} = -\frac{1}{2}\left(1 + \frac{\lambda^2 - 1}{\lambda^2 + 1}\cos 2\psi\right)$$

$$+\frac{\mathrm{St}}{2E_1(\lambda)}\frac{(\lambda^2 - 1)}{(\lambda^2 + 1)^2}[1+(\lambda^2+(\lambda^2-1)\cos 2\psi)\sin^2\theta]\sin 2\psi + O(\mathrm{St}^2), \tag{12}$$

$$\dot{\theta} = \frac{1 - \lambda^2}{4(1 + \lambda^2)}\sin 2\psi \sin 2\theta$$

$$+\frac{\mathrm{St}}{4E_1(\lambda)}\frac{(\lambda^2 - 1)}{(\lambda^2 + 1)^2}[1 + \lambda^2 + (\lambda^2 - 1)\cos 2\psi + (1 - \lambda^2)\cos 2\psi \cos 2\theta$$

$$+(1 - \lambda^2)\cos 2\theta]\sin^2\psi \sin 2\theta + O(\mathrm{St}^2) \tag{13}$$

# Comparison between asymptotic and exact Eqs.

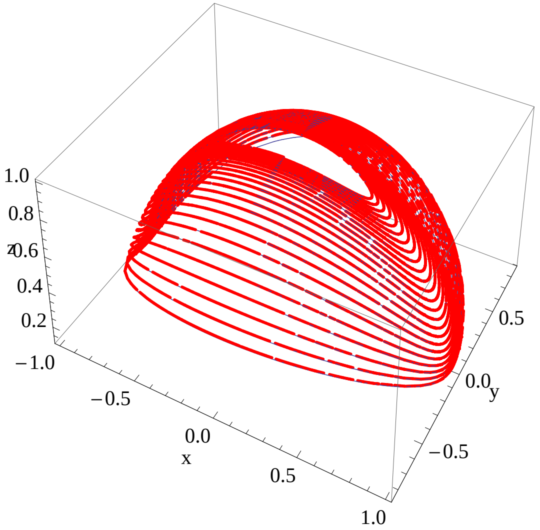



Figure: St = 0.8 and $\lambda = 5$.

# Comparison between asymptotic and exact Eqs.

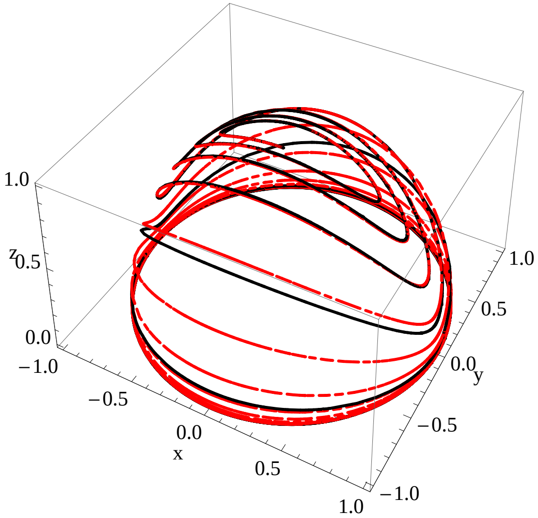



Figure: St = 4 and $\lambda = 5$.

# Floquet exponents of periodic orbits

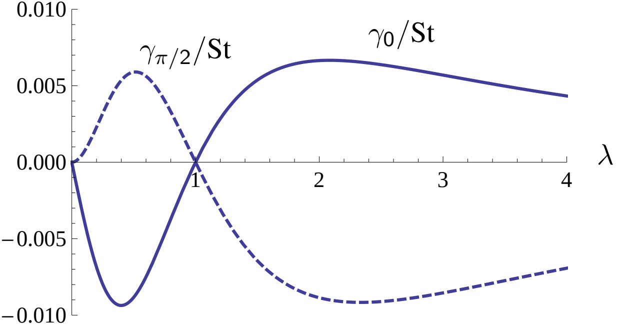
- Floquet exponent of the logroll (i.e. "polar") orbit $\theta = 0$ is obtained from Eqs. (12) and (13):

$$\frac{\theta(nT)}{\theta(0)} = \exp(\gamma_0 \times nT), \text{with} \quad \gamma_0 = \frac{1}{2}\frac{(\lambda^2 - 1)}{(\lambda^2+1)^2 E_1(\lambda)} \mathrm{St} \qquad (14)$$

- Floquet exponent of the equatorial orbit $\theta = \pi/2$ is obtained from Eqs. (12) and (13):

$$\frac{\theta(nT) - \pi/2}{\theta(0) - \pi/2} = \exp(\gamma_{\pi/2} \times nT), \text{with} \quad \gamma_{\pi/2} = \frac{\lambda(1-\lambda)}{(\lambda^2+1)^2 E_1(\lambda)} \mathrm{St} \qquad (15)$$

# Floquet exponents of periodic orbits




$$\gamma_0 = \frac{1}{2}\frac{(\lambda^2-1)}{(\lambda^2+1)^2\,E_1(\lambda)}\,\mathrm{St}\,, \quad \gamma_{\pi/2} = \frac{\lambda(1-\lambda)}{(\lambda^2+1)^2\,E_1(\lambda)}\,\mathrm{St} \qquad (16)$$

# Floquet exponents of periodic orbits

- Points: $\ln[(\theta(nT) - \theta_{eq})/(\theta(0) - \theta_{eq})]/nT$.
- Lines: theoretical Floquet exponents.

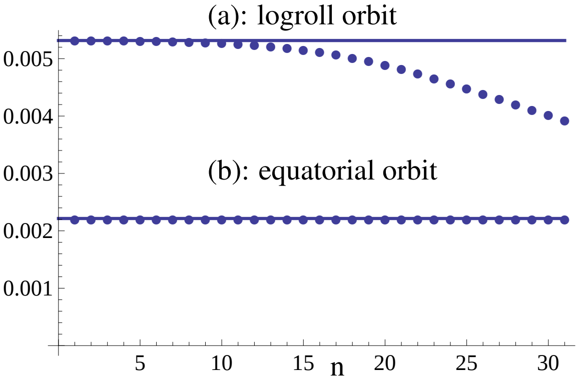



Figure: St = 0.8 in case (a) and 0.05 in case (b).

# Effect of inertia and rotational noise

$$\frac{d\vec{n}}{dt} = \vec{\Omega} \times \vec{n} + \vec{\omega} \times \vec{n}$$

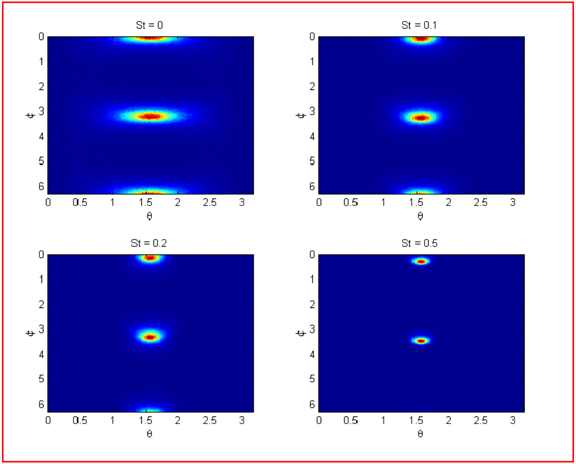
with

$$\vec{\Omega} = \vec{F}_0(\vec{X}) + \mathrm{St}\, \vec{F}_1(\vec{X}) + O(\mathrm{St}^2).$$

and:

$$< w_i(t) w_j(t') >= \frac{2}{\mathrm{Pe}} \delta_{ij} \delta(t - t')$$

# Effect of inertia and rotational noise




$\lambda = 5$, Pe $= 1000$

# Conclusion

- Asymptotic formulation in the limit of weak inertia shows the expected behavior of rods and disks.

- Enables to derive Floquet exponents showing the tendency to quit Jeffery orbits and converge towards short axis aligned with vorticity (confirm Lundell & Carlsson 2010, Lundell 2011).

- Enables a straightforward formulation for particles sensitive to both noise and inertia.

# Euler angles