Dynamics of a dumbbell in linear flows by the method of reflexions

Fabien Candelier

IUSTI (UMR CNRS 7343)
Université d'Aix-Marseille

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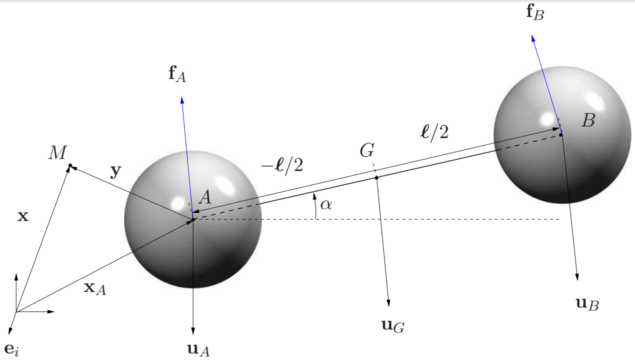
Introduction and notations



Introduction: The interest of studying dumbbells lies in the simplicity of their geometries $\Rightarrow$ exploitation of the many results available on spheres.

Basic assumptions: The dumbbell is composed of two identical spheres, denoted by *A* and *B*, of mass *m* and radius *a*, linked by a virtual rigid-rod.

Angular momentum of the dumbbell

Orientational dynamics is governed by the angular momentum equation

$$\dot{\boldsymbol{\sigma}}_G = \mathbf{m}_h \tag{1}$$

where

- $\boldsymbol{\sigma}_G = m \left(\left(-\frac{\boldsymbol{\ell}}{2} \times \mathbf{u}_A + \frac{2}{5}\, a^2\, \boldsymbol{\omega} \right) + \left(\frac{\boldsymbol{\ell}}{2} \times \mathbf{u}_B + \frac{2}{5}\, a^2\, \boldsymbol{\omega} \right) \right)$,
- $\boldsymbol{\omega}$ is the angular velocity of the body,
- $\mathbf{m_h}$ is the hydrodynamic torque acting on the spheres.

By exploiting (i) the rigid motion of the dumbbell

$$\mathbf{u}_B = \mathbf{u}_G + \boldsymbol{\omega} \times \frac{\boldsymbol{\ell}}{2} \quad \text{and} \quad \mathbf{u}_A = \mathbf{u}_G - \boldsymbol{\omega} \times \frac{\boldsymbol{\ell}}{2} .$$

and (ii) by introducing the orthogonal vectors $\mathbf{t}_1$, $\mathbf{t}_2$ and $\mathbf{t}_3$ such that $\boldsymbol{\ell} = \ell\, \mathbf{t}_1$,

$$\dot{\boldsymbol{\sigma}}_G = \frac{m\ell^2}{2} \left(\left(1 + \frac{8}{5} \frac{a^2}{\ell^2} \right) \mathbf{t}_1 \times \ddot{\mathbf{t}}_1 + \frac{8}{5} \frac{a^2}{\ell^2}\, \dot{\Omega}^1\, \mathbf{t}_1 + \frac{8}{5} \frac{a^2}{\ell^2}\, \Omega^1\, \dot{\mathbf{t}}_1 \right) .$$

Hydrodynamic torque

The hydrodynamic torque acting on the dumbbell

$$\mathbf{m}_h = \frac{\boldsymbol{\ell}}{2} \times \delta\mathbf{f} + \mathbf{m}_{hA} + \mathbf{m}_{hB} \ ,$$

- $\mathbf{m}_{hA}$ and $\mathbf{m}_{hB}$ are the torque acting on the spheres A and B (w.r.t. their centres)
- $\delta\mathbf{f} = \mathbf{f}_B - \mathbf{f}_A$

1) Projection of the angular momentum eq. along $\mathbf{t}_1$ (spin equation):

$$m\frac{4}{5}\, a^2\, \dot{\Omega}^1 = (\mathbf{m}_{hA} + \mathbf{m}_{hB}) \cdot \mathbf{t}_1$$

2) Cross product the angular momentum eq. with $\mathbf{t}_1$:

$$m\ell\left(1 + \frac{8}{5}\frac{a^2}{\ell^2}\right)\left(\ddot{\mathbf{t}}_1 + (\dot{\mathbf{t}}_1 \cdot \dot{\mathbf{t}}_1)\,\mathbf{t}_1\right) = \left(\mathbf{I} - \mathbf{t}_1 \otimes \mathbf{t}_1\right) \cdot \delta\mathbf{f} + \frac{2}{\ell}(\mathbf{m}_{hA} + \mathbf{m}_{hB}) \times \mathbf{t}_1 + m\ell\,\frac{8}{5}\frac{a^2}{\ell^2}\Omega^1\,\mathbf{t}_1 \times \dot{\mathbf{t}}_1 \ . \tag{2}$$

Angular momentum equation

It is worth noting that the tensor

$$\left(\mathbf{I} - \mathbf{t}_1 \otimes \mathbf{t}_1\right) = \left(\mathbf{t}_2 \otimes \mathbf{t}_2 + \mathbf{t}_3 \otimes \mathbf{t}_3\right) = \mathbf{P}_\perp$$

actually defines a projector onto the plane $(\mathbf{t}_2, \mathbf{t}_3)$.
By introducing explicitly the components of the vector $\dot{\mathbf{t}}_1$ (which is $\perp$ to $\mathbf{t}_1$)

$$\dot{\mathbf{t}}_1 = V^2\,\mathbf{t}_2 + V^3\,\mathbf{t}_3 = V^\alpha\,\mathbf{t}_\alpha \quad (V^\alpha \equiv \text{ Angular velocity})$$

it may be shown after some simple algebra, that

$$\ddot{\mathbf{t}}_1 + (\dot{\mathbf{t}}_1 \cdot \dot{\mathbf{t}}_1)\,\mathbf{t}_1 = \dot{V}^2\,\mathbf{t}_2 + \dot{V}^3\,\mathbf{t}_3 + \Omega^1\mathbf{t}_1 \times \dot{\mathbf{t}}_1\ .$$

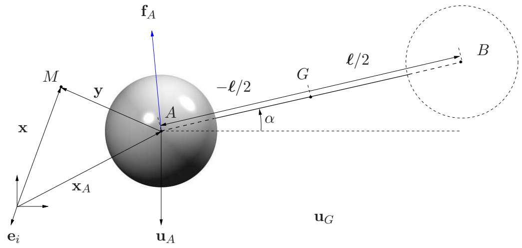
As a result the equation governing the orientational dynamics simply reads as

$$\left(1 + \frac{8}{5}\frac{a^2}{\ell^2}\right)\dot{V}^\alpha\mathbf{t}_\alpha = \frac{\mathbf{P}_\perp \cdot \delta\mathbf{f}}{m\ell} + \frac{2}{m\ell^2}\left((\mathbf{m}_{hA} + \mathbf{m}_{hB}) \times \mathbf{t}_1\right) - \left(\Omega^1\mathbf{t}_1 \times \dot{\mathbf{t}}_1\right), \quad \alpha = 2, 3.$$

$\hookrightarrow$ the problem is the determination of $\delta\mathbf{f}$

Introduction of the method of reflexions in a simple case

The method of reflexions is based on an iterative process:



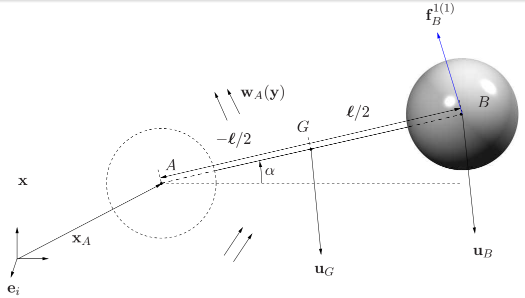
- As a first step, the sphere A is considered as if it were alone in fluid

$$\mathbf{w}_A(\mathbf{y}) = -\mathbf{G}(\mathbf{y}) \cdot \mathbf{f}_A^{1(0)} \quad \text{where} \quad \mathbf{f}_A^{1(0)} = -6\pi\mu a\, \mathbf{u}_A$$

and $\mathbf{G}$ is a tensor whose components (in the Cartesian basis) are

$$G_{ij} = \frac{1}{8\pi\mu}\left(\frac{\delta_{ij}}{r} + \frac{y_i y_j}{r^3}\right), \qquad r = |\mathbf{y}| .$$

Introduction of the method of reflexions



- As a second step, the sphere B is introduced in the flow field $\mathbf{w}_A$, and the force acting on it reads as (Faxen's corrections $\sim O(a^3/\ell^3)$)

$$\mathbf{f}_B^{1(1)} = -6\pi\mu a(\mathbf{u}_B - \mathbf{w}_A(\ell)) ,$$

$$\hookrightarrow \quad \mathbf{w}_B = -\mathbf{G}\cdot\mathbf{f}_B^{1(1)} = 6\pi\mu a\,(\mathbf{G}\cdot\mathbf{u}_B - 6\pi\mu a\,\mathbf{G}\cdot\mathbf{G}\cdot\mathbf{u}_A) ,$$

Introduction of the method of reflexions

- and so on... Force acting on the sphere A corrected up to $O(a^2/\ell^2)$:

$$\mathbf{f}_A^{1(2)} = -6\pi\mu a(\mathbf{u}_A - 6\pi\mu a\mathbf{G}(-\boldsymbol{\ell})\cdot\mathbf{u}_B + (6\pi\mu a)^2\mathbf{G}(-\boldsymbol{\ell})\cdot\mathbf{G}(-\boldsymbol{\ell})\cdot\mathbf{u}_A) + O(a^3/\ell^3).$$

and reciprocally,

$$\mathbf{f}_B^{1(2)} = -6\pi\mu a(\mathbf{u}_B - 6\pi\mu a\mathbf{G}(\boldsymbol{\ell})\cdot\mathbf{u}_A + (6\pi\mu a)^2\mathbf{G}(\boldsymbol{\ell})\cdot\mathbf{G}(\boldsymbol{\ell})\cdot\mathbf{u}_B) + O(a^3/\ell^3).$$

By using the following property: $\mathbf{G}(\boldsymbol{\ell}) = \mathbf{G}(-\boldsymbol{\ell})$,

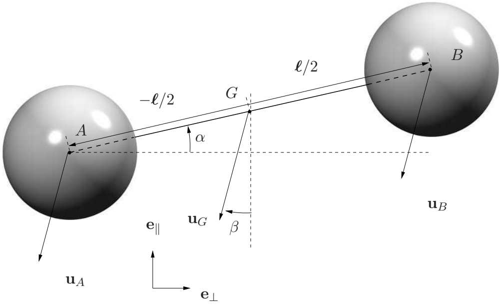
$$\delta\mathbf{f} = -6\pi\mu a\Big(\mathbf{I} + 6\pi\mu a\,\mathbf{G}(\boldsymbol{\ell}) + (6\pi\mu a)^2\,\mathbf{G}(\boldsymbol{\ell})\cdot\mathbf{G}(\boldsymbol{\ell})\Big)\cdot\dot{\boldsymbol{\ell}} + O(a^3/\ell^3)\,.$$

In this simple Stokes problem, it is finally found that

$$\mathbf{P}_\perp\cdot\delta\mathbf{f} = -6\pi\mu a\left(1 + \frac{3}{4}\frac{a}{\ell} + \frac{9}{16}\left(\frac{a}{\ell}\right)^2\right)V^\alpha\mathbf{t}_\alpha\,.$$

As expected in the case $\boldsymbol{\omega} = 0$ at $t = 0$, i.e. $\dot{\mathbf{t}}_1 = 0$, the angular velocity remains zero at any time.

Results in a quiescent fluid



- Solving the (normalized) momentum equation of the dumbbell yields

$$\mathbf{u}_G \sim -\left(\mathbf{I} - 6\pi\,\mathbf{G}(\ell)\right)^{-1} \cdot \mathbf{e}_3\,, \quad \text{and} \quad \beta = \arctan\left(\frac{3\cos(\alpha)\sin(\alpha)}{4\ell - 3 - 3\cos(\alpha)^2}\right).$$

- β angle of the trajectory w.r.t. the vertical

Method of reflexions in linear flows

We consider now the case where the dumbbell is immersed in a linear flow

$$\mathbf{v} = \mathbf{A} \cdot \mathbf{x} \,,$$

- **Unperturbed force** acting the sphere A and B are not identical:

$$\delta \mathbf{f}^0 = m_f \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}\Big|_{\mathbf{x}_B} - m_f \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}\Big|_{\mathbf{x}_A} = m_f \, \mathbf{A}^2 \cdot \boldsymbol{\ell} \,.$$

- **Perturbation force:** very similar results are found except that the velocities $\mathbf{u}_A$ and $\mathbf{u}_B$ have to be replaced here by the relative (slip) velocities $\mathbf{u}_A - \mathbf{v}(\mathbf{x}_A)$ and $\mathbf{u}_B - \mathbf{v}(\mathbf{x}_B)$.

By using the fact $\mathbf{v}(\mathbf{x}_B) - \mathbf{v}(\mathbf{x}_A) = \mathbf{A} \cdot \boldsymbol{\ell}$

$$\delta \mathbf{f}^1 = -6\pi\mu a \Big(\mathbf{I} + 6\pi\mu a \, \mathbf{G}(\boldsymbol{\ell}) + (6\pi\mu a)^2 \, \mathbf{G}(\boldsymbol{\ell}) \cdot \mathbf{G}(\boldsymbol{\ell}) \Big) \cdot (\dot{\boldsymbol{\ell}} - \mathbf{A} \cdot \boldsymbol{\ell}) \,,$$

and finally, we are led to

$$\delta \mathbf{f} = \delta \mathbf{f}^0 + \delta \mathbf{f}^1 = m_f \mathbf{A}^2 \cdot \boldsymbol{\ell} - 6\pi\mu a \Big(\mathbf{I} + 6\pi\mu a \mathbf{G}(\boldsymbol{\ell}) + (6\pi\mu a)^2 \mathbf{G}(\boldsymbol{\ell}) \cdot \mathbf{G}(\boldsymbol{\ell}) \Big) \cdot (\dot{\boldsymbol{\ell}} - \mathbf{A} \cdot \boldsymbol{\ell}) .$$

Method of reflexions in linear flows

For the hydrodynamic torques (which scale as $O(a^2/\ell^2)$) no reflexions are needed.
By normalising time with $1/\sqrt{\mathbf{A}:\mathbf{A}}$, lengths by a, and by introducing the classical decomposition

$$\mathbf{A} = \mathbf{\Omega}_f + \mathbf{E} \quad \text{where} \quad \mathbf{E} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^t\right) \quad \text{and} \quad \mathbf{\Omega}_f = \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^t\right)$$

we are led to

$$\left(1 + \frac{8}{5}\frac{a^2}{\ell^2}\right)\, \dot{V}^\alpha\, \mathbf{t}_\alpha = \gamma\, \mathbf{P}_\perp \cdot \mathbf{A}^2 \cdot \mathbf{t}_1 - \frac{1}{S_t}\left(\frac{9}{2} + \frac{27}{8}\frac{a}{\ell} + \frac{849}{32}\frac{a^2}{\ell^2}\right)(\dot{\mathbf{t}}_1 - \mathbf{\Omega}_f \cdot \mathbf{t}_1)$$
$$+ \frac{1}{S_t}\left(\frac{9}{2} + \frac{27}{8}\frac{a}{\ell} + \frac{81}{32}\frac{a^2}{\ell^2}\right)\mathbf{P}_\perp \cdot (\mathbf{E} \cdot \mathbf{t}_1)\,. \tag{3}$$

where

$$\gamma = \frac{\rho_f}{\rho_p} \quad \text{and} \quad S_t = \frac{a^2}{\nu\,\tau}\frac{\rho_p}{\rho_f} \quad \text{(Stokes number).}$$

Recovering Jeffery's orbits

In the limit where $1/S_t \to \infty$,

$$\dot{\mathbf{t}}_1 = \boldsymbol{\Omega}_f \cdot \mathbf{t}_1 + \frac{\frac{9}{2} + \frac{27}{8}\frac{a}{\ell} + \frac{81}{32}\frac{a^2}{\ell^2}}{\frac{9}{2} + \frac{27}{8}\frac{a}{\ell} + \frac{849}{32}\frac{a^2}{\ell^2}} \mathbf{P}_\perp \cdot (\mathbf{E} \cdot \mathbf{t}_1) \, .$$

To simplify this equation, we may note that

$$\frac{\frac{9}{2} + \frac{27}{8}\frac{a}{\ell} + \frac{81}{32}\frac{a^2}{\ell^2}}{\frac{9}{2} + \frac{27}{8}\frac{a}{\ell} + \frac{849}{32}\frac{a^2}{\ell^2}} \sim 1 - \frac{16}{3}\frac{a^2}{\ell^2} + O\left(\frac{a^3}{\ell^3}\right) \, ,$$

so that a dumbbell of a given aspect ration (i.e. ℓ/a) should have the very same behaviour of an ellipsoid whose aspect ratio is given by

$$r \sim \frac{\sqrt{6}}{4}\frac{\ell}{a} \, .$$

(see Hinch & Leal 1973)

Inertia effects in a quiescent fluid (Khayat & Cox 1989)

We denote by

$$\boldsymbol{\epsilon} = \frac{a\mathbf{u}}{\nu}$$

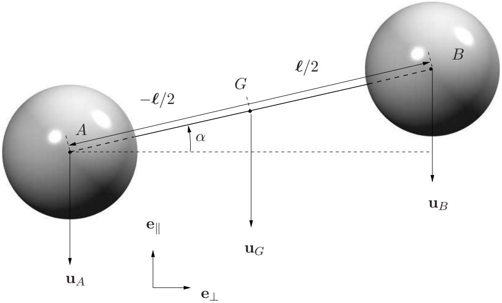
the (vectorial) Reynolds number of the sphere, so that the Oseen's equations to solve are

$$\begin{aligned} -\boldsymbol{\epsilon} \cdot \boldsymbol{\nabla} \mathbf{w} &= -\boldsymbol{\nabla} p + \nabla^2 \mathbf{w} + \mathbf{f}\,\delta\,, & (4) \\ \boldsymbol{\nabla} \cdot \mathbf{w} &= 0\,. & (5) \end{aligned}$$

The Green's function of the Oseen's equation (found by using Fourier Transforms):

$$\mathbf{w} = \frac{\exp\Big(-\frac{1}{2}(\epsilon r + \boldsymbol{\epsilon} \cdot \mathbf{y})\Big)}{8\pi\, r}\mathbf{f} + \left(1 - \left(1 + \frac{\epsilon\, r}{2}\right) \exp\left(-\frac{1}{2}(\epsilon r + \boldsymbol{\epsilon} \cdot \mathbf{y})\right)\right) \frac{f\,\mathbf{y}}{\epsilon 4\pi\, r^3}\,. \quad (6)$$

Inertia effects in quiescent fluid (Khayat & Cox 1989)



In the limit where $\mathbf{u}_B - \mathbf{u}_A \sim O(\mathrm{Re})$: $\delta\mathbf{f} = 6\pi\left(\mathbf{w}(\ell) - \mathbf{w}(-\ell)\right)$

$$\delta\mathbf{f} = -\frac{3}{8}\mathrm{Re}\left(\sin\alpha(1+\sin^2\alpha)\,\mathbf{e}_{\|} - \cos^3\alpha\,\mathbf{e}_{\perp}\right) \quad \Leftrightarrow \quad \mathbf{P}_{\perp}\cdot\delta\mathbf{f} = -\frac{3}{8}\mathrm{Re}\sin 2\alpha\,\mathbf{t}_{\perp}$$

$\hookrightarrow$ Equilibrium: $\alpha = 0$ (stable) and $\alpha = \pi/2$ (unstable)

Inertia effects in Linear flows

In a linear flow field, the (steady) perturbation flow produced can be expanded (in a region defined by $r \sim a/\mathrm{Re}^{1/2}$) in the form

$$\mathbf{w}_A = -\mathbf{G} \cdot \mathbf{f}_A^{1(0)} - \mathrm{Re}^{1/2}\mathbf{M} \cdot (\mathbf{u}_A - \mathbf{v}(\mathbf{x}_A))$$

i.e. **Stokeslet + a uniform flow** (fluid inertia effects).

If we assume that $\ell \sim a/\mathrm{Re}^{1/2} \Rightarrow$ the sphere B is (i) located in the far-field flow produced by the sphere A, and (ii) submitted to inertia effects:

$$\mathbf{f}_B^{1(1')} = -6\pi\left(\mathbf{I} + \mathrm{Re}^{1/2}\,\mathbf{M}\right) \cdot \left(\mathbf{u}_B - 6\pi\mathbf{G}(\ell) \cdot \mathbf{u}_A + \mathrm{Re}^{1/2}\mathbf{M} \cdot \mathbf{u}_A\right) .$$

Pursuing the iterations up to $O(a^3/\ell^3)$ provides us with

$$\mathbf{f}_A^{1(2')} = -6\pi\left(\mathbf{I}+\mathrm{Re}^{1/2}\mathbf{M}\right)\cdot\left(\mathbf{u}_A-6\pi\mathbf{G}(\ell)\cdot\mathbf{u}_B+\mathrm{Re}^{1/2}\mathbf{M}\cdot\mathbf{u}_B+(6\pi)^2\mathbf{G}(\ell)\cdot\mathbf{G}(\ell)\cdot\mathbf{u}_A\right)$$

where the last two terms are of the same order of magnitude $O(a^2/\ell^2)$.

Fluid inertia effects in quiescent

We are finally led to

$$\delta \mathbf{f}^{1} = -6\pi\Big(\mathbf{I}+6\pi\mathbf{G}(\ell)+(6\pi)^{2}\mathbf{G}(\ell)\cdot\mathbf{G}(\ell)+6\pi\mathrm{Re}^{1/2}\mathbf{M}\cdot\mathbf{G}(\ell)-\mathrm{Re}\mathbf{M}\cdot\mathbf{M}\Big)\cdot(\dot{\ell}-\mathbf{A}\cdot\ell)\ :,$$

Note that in the case of a rotating fluid the components of **M** in the Cartesian basis (Herron *et a.* 1975)

$$\mathbf{M} = \begin{pmatrix} 5/7 & -3/5 & 0 \\ 3/5 & 5/7 & 0 \\ 0 & 0 & 4/7 \end{pmatrix} .$$

Similarly, in the case of a pure shear flow $\mathbf{A} = \mathbf{e}_1 \otimes \mathbf{e}_3$ (Miyazaki *et al.* 1995)

$$\mathbf{M} = \begin{pmatrix} 0.0743 & 0 & 0.944 \\ 0 & -0.577 & 0 \\ 0.343 & 0 & 0.327 \end{pmatrix} .$$

Problem in the case of a pure shear flow...

Conclusions

- Many results concerning the behaviours of fibres are well recovered with the dumbbell (sedimentation in a fluid at rest including particle Reynolds number effects, Jeffery's orbits).
- Using the method of reflexions seems promising to investigate fluid inertia effects on the orientational dynamics of dumbbells.
- Such results should provide us with correct tendencies concerning fluid inertia on fibres.
- However, in general, the perturbation flow produced by the sphere is affected both by convective inertia effect and unsteady effect.

  $\hookrightarrow$ Taking both these effects into account remains a challenging task.