Triaxial particles in shear flows : a Floquet analysis using a two-vector formulation

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Introduction

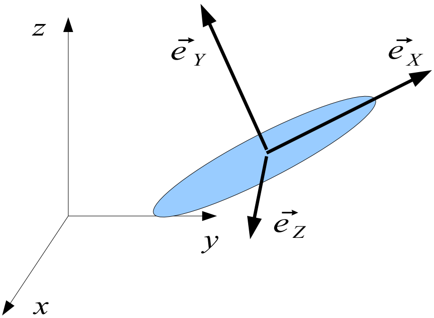
• Many flows carry tiny solid objects with a very small particle Reynolds number (inertia-free limit).

• Geophysical flows : aerosols, sediments, etc.

• Industrial flows : chemical engineering, combustion, etc.

• In many cases particles are non-spherical: rods, platelets, ...complex orientational dynamics.

Axisymmetric ellipsoid or dumbbell in a simple shear flow



- In the absence of inertia and brownian motion, these particles have periodic orientations (Jeffery 1922).

- **Goal:**
Investigate the stability of some orbits when triaxiality is present.
(See also Hinch & Leal 1979 ; Yarin et al. 1997).

Motion equations

- Tiny ellipsoid with semi-axes a, b, c, in flow $\vec{u}$.
- Angular velocity vector with respect to the laboratory frame: $\vec{\Omega}(t)$
- Its components in a cartesian basis attached to the object:
$$\Omega_X, \Omega_Y, \Omega_Z$$
- Jeffery 1922 (inertia-free object):

$$\Omega_X = \frac{b^2 g_{ZY} - c^2 g_{YZ}}{b^2 + c^2}, \quad (1)$$

$$\Omega_Y = \frac{c^2 g_{XZ} - a^2 g_{ZX}}{c^2 + a^2}, \quad (2)$$

$$\Omega_Z = \frac{a^2 g_{YX} - b^2 g_{XY}}{a^2 + b^2} \quad (3)$$

where $\mathbf{g} = \nabla \vec{u}$

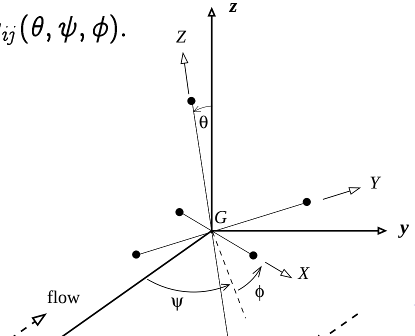
The orientational dynamics is extremely complex: periodic motion, quasi-periodic, chaotic (Yarin et al. 1997).

$$\dot\psi \sin\theta \sin\phi + \dot\theta \cos\phi = \frac{b^2 g_{ZY} - c^2 g_{YZ}}{b^2 + c^2}, \quad (4)$$

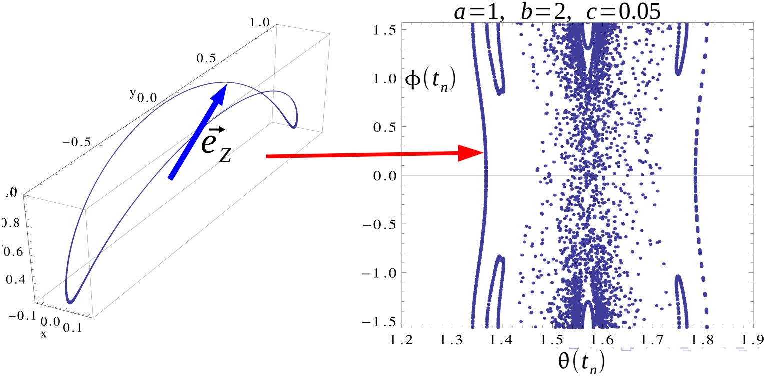
$$\dot\psi \sin\theta \cos\phi - \dot\theta \sin\phi = \frac{c^2 g_{XZ} - a^2 g_{ZX}}{c^2 + a^2}, \quad (5)$$

$$\dot\psi \cos\theta + \dot\phi = \frac{a^2 g_{YX} - b^2 g_{XY}}{a^2 + b^2} \quad (6)$$

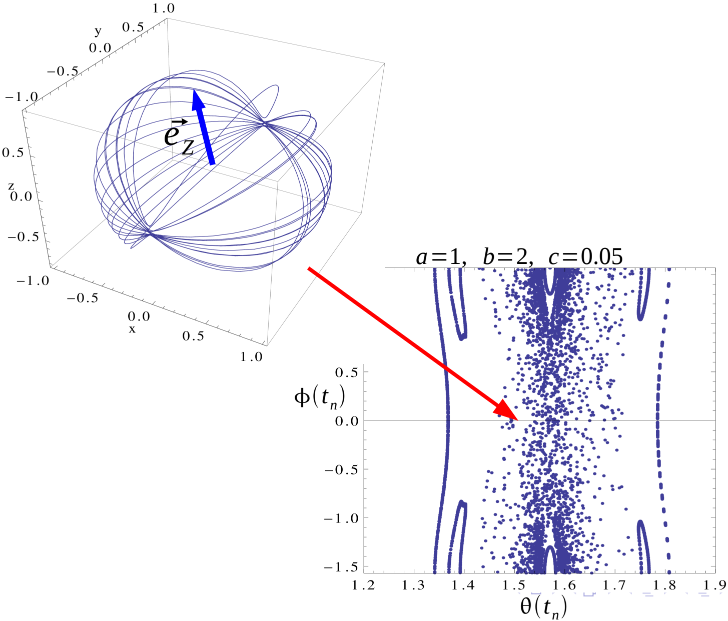
with $g_{ij} = g_{ij}(\theta, \psi, \phi)$.



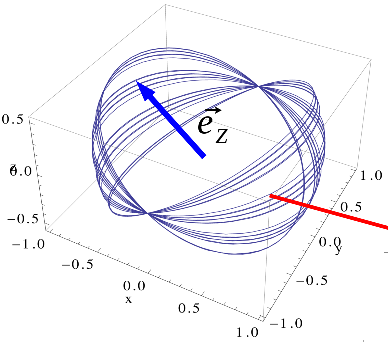
Complex dynamics of non-spherical objects: examples

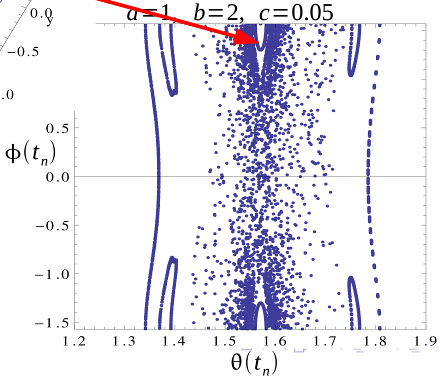


Complex dynamics of non-spherical objects: examples



Complex dynamics of non-spherical objects: examples



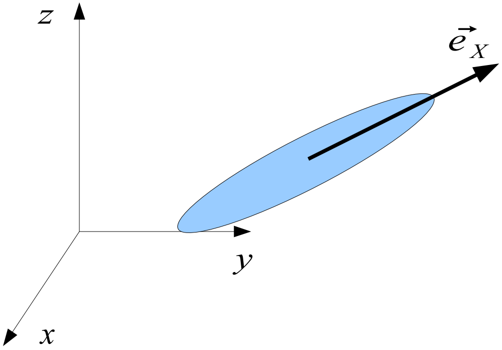


Goal: re-visit the dynamics in vector form for triaxial objects.

Axisymmetric object ($c = b$):

$$\dot{\vec{e}}_X = \mathbf{B}\vec{e}_X - (\vec{e}_X.\mathbf{B}\vec{e}_X)\vec{e}_X$$

where $\mathbf{B} = (a^2\nabla\vec{u} - c^2\nabla\vec{u}^T)/(a^2 + c^2)$ (Hinch & Leal 1971).



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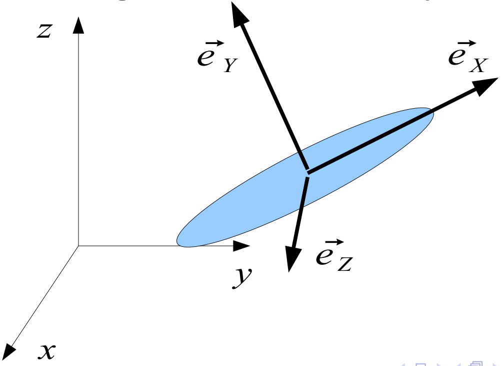
$\rightarrow$ to be generalized to triaxial objects.



The dynamics in vector form for triaxial objects.

Starting from the rotation vector of non-inertial triaxial object, and since $\dot{\vec{e}}_X = \vec{\Omega} \times \vec{e}_X$:

$$\Omega_Y = \frac{c^2 g_{XZ} - a^2 g_{ZX}}{c^2 + a^2} = -\dot{\vec{e}}_X . \vec{e}_Z, \tag{7}$$

$$\Omega_Z = \frac{a^2 g_{YX} - b^2 g_{XY}}{a^2 + b^2} = \dot{\vec{e}}_X . \vec{e}_Y \tag{8}$$

Leads to:

$$\dot{\vec{e}}_X = \frac{a^2 g_{YX} - b^2 g_{XY}}{a^2 + b^2} \vec{e}_Y - \frac{c^2 g_{XZ} - a^2 g_{ZX}}{c^2 + a^2} \vec{e}_Z$$

Eliminate $g_{XZ} \vec{e}_Z = \nabla \vec{u}^T \vec{e}_X - g_{XX} \vec{e}_X - g_{XY} \vec{e}_Y$:

The dynamics in vector form for triaxial objects.

Leads to:

$$\left(\mathbf{S} = (\nabla\vec{u} + \nabla\vec{u}^T)/2\right)$$

$$\boxed{\dot{\vec{e}}_X = \underbrace{f_1\,(\vec{e}_Y.\mathbf{S}\vec{e}_X)\vec{e}_Y}_{\text{"}triaxial\ \ term\text{"}} + \mathbf{B}\vec{e}_X - (\vec{e}_X.\mathbf{B}\vec{e}_X)\vec{e}_X}$$

where $\mathbf{B} = (a^2\nabla\vec{u} - c^2\nabla\vec{u}^T)/(a^2 + c^2)$

$$\text{and } f_1 = \frac{2a^2(c^2 - b^2)}{(a^2 + b^2)(a^2 + c^2)}.$$

Same treatment for $\dot{\vec{e}}_Y$:

$$\boxed{\dot{\vec{e}}_Y = \underbrace{f_2\,(\vec{e}_Y.\mathbf{S}\vec{e}_X)\vec{e}_X}_{\text{"}triaxial\ \ term\text{"}} + \mathbf{C}\vec{e}_Y - (\vec{e}_Y.\mathbf{C}\vec{e}_Y)\vec{e}_Y}$$

where $\mathbf{C} = (b^2\nabla\vec{u} - c^2\nabla\vec{u}^T)/(b^2 + c^2)$

$$\text{and } f_2 = \frac{2b^2(c^2 - a^2)}{(a^2 + b^2)(b^2 + c^2)}.$$

The dynamics in vector form for triaxial objects.

If

$$\vec{e}_X = \alpha(t) \underbrace{\vec{e}_x}_{\in Lab.\ Frame} + \beta(t)\vec{e}_y + \gamma(t)\vec{e}_z$$

$$\vec{e}_Y = \delta(t) \underbrace{\vec{e}_x}_{\in Lab.\ Frame} + \eta(t)\vec{e}_y + \xi(t)\vec{e}_z$$

then the dynamics in the plane of shear $(\alpha, \beta, \delta, \eta)$ is independent of the the z (vorticity) coordinates (γ, ξ).

4 degrees of freedom (not minimal) with cubic non-linearities.

The dynamics in vector form for triaxial objects.

Set:

$$\begin{aligned}
\vec{e}_X &= \alpha(t)\vec{e}_x + \beta(t)\vec{e}_y + \gamma(t)\vec{e}_z, &(9)\\
\vec{e}_Y &= \delta(t)\vec{e}_x + \eta(t)\vec{e}_y + \xi(t)\vec{e}_z &(10)
\end{aligned}$$

then, the exact triaxial inertia-free dynamics is:

$$\begin{aligned}
\dot{\alpha} &= \frac{a^2}{a^2+c^2}\beta - \frac{a^2-c^2}{a^2+c^2}\alpha^2\beta + \frac{f_1}{2}(\delta\beta+\eta\alpha)\delta, &(11)\\
\dot{\beta} &= -\frac{c^2}{a^2+c^2}\alpha - \frac{a^2-c^2}{a^2+c^2}\alpha\beta^2 + \frac{f_1}{2}(\delta\beta+\eta\alpha)\eta, &(12)\\
\dot{\delta} &= \frac{b^2}{b^2+c^2}\eta - \frac{b^2-c^2}{b^2+c^2}\delta^2\eta + \frac{f_2}{2}(\delta\beta+\eta\alpha)\alpha, &(13)\\
\dot{\eta} &= \underbrace{-\frac{c^2}{b^2+c^2}\delta - \frac{b^2-c^2}{b^2+c^2}\delta\eta^2}_{\text{periodic dynamics (J. orbits)}} + \frac{f_2}{2}(\delta\beta+\eta\alpha)\beta, &(14)
\end{aligned}$$

APPLICATION: stability of orbit $Y = z$, for all a, b, c

Set:

$$\vec{e}_X(t) := \vec{N}_0(t) + \vec{N}_1(t) \qquad and \qquad \vec{e}_Y(t) := \vec{e}_z + \vec{P}_1(t)$$

With: $\dot{\vec{N}}_0 = \mathbf{B}\vec{N}_0 - (\vec{N}_0.\mathbf{B}\vec{N}_0)\vec{N}_0$ (Jeffery orbit), then:

$$\dot{\vec{N}}_1 = -(\vec{N}_0.\mathbf{B}\vec{N}_0)\vec{N}_1 + f_1(\vec{P}_1.\mathbf{S}\vec{N}_0)\vec{e}_z + (quad.\, terms), \tag{15}$$

$$\dot{\vec{P}}_1 = \mathbf{C}\vec{P}_1 + f_2(\vec{P}_1.\mathbf{S}\vec{N}_0)\vec{N}_0 + (quad.\, terms) \tag{16}$$

are the departure from the J. orbit.

APPLICATION: stability of orbit $Y = z$, for all a, b, c

Finally:

$$\vec{e}_X(t) := \vec{N}_0(t) + \vec{N}_1(t) \qquad \textit{and} \qquad \vec{e}_Y(t) := \vec{e}_z + \vec{P}_1(t)$$

$$\dot{\vec{N}}_1 = -(\vec{N}_0.\mathbf{B}\vec{N}_0)\vec{N}_1 + f_1(\vec{P}_1.\mathbf{S}\vec{N}_0)\vec{e}_z, \tag{17}$$

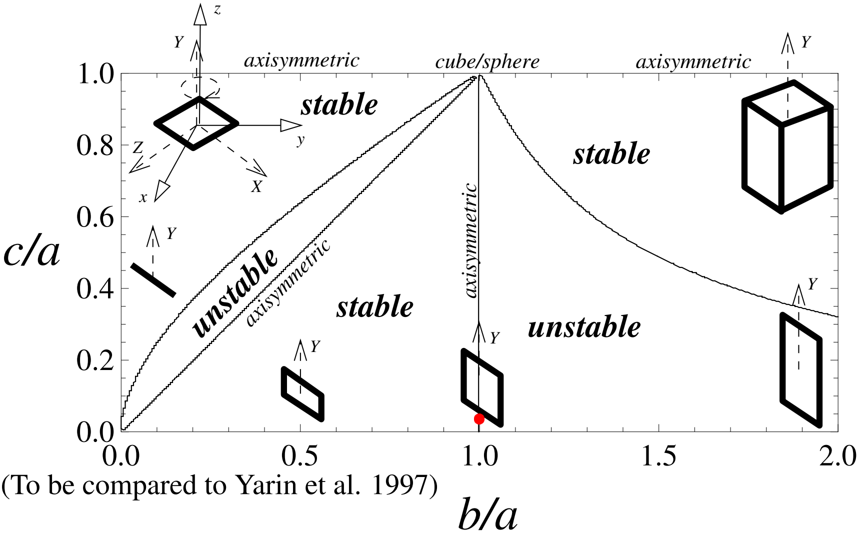
$$\dot{\vec{P}}_1 = \mathbf{C}\vec{P}_1 + f_2(\vec{P}_1.\mathbf{S}\vec{N}_0)\vec{N}_0 \tag{18}$$

- Of the form $\dot{\vec{q}} = \mathbf{A}(t)\vec{q}$ with $\mathbf{A}(t) = \mathbf{A}(t+T)$

- Floquet theory: $\vec{q}(t) = \sum_i C_i e^{\mu_i t}\vec{p}_i(t)$, where p_i's are T-periodic and $\exp(\mu_i T)$ are the eigenvalues of the monodromy matrix $\mathbf{M}(T)$ [i.e. $\dot{\mathbf{M}} = \mathbf{A}(t).\mathbf{M}(t)$, $\quad \mathbf{M}(0) = \mathbf{I_d}$.]

$\rightarrow$ $Re(\mu_i)$, if non-zero, indicates stability/instability of the orbit.

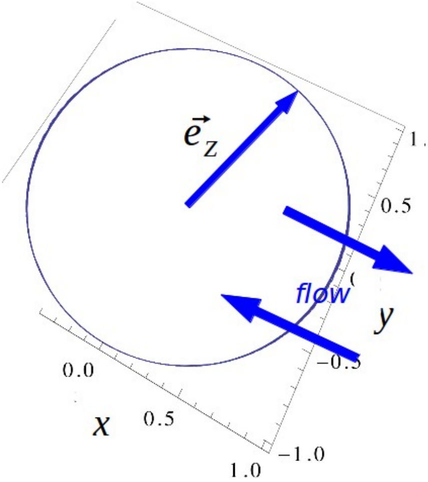
Result: Floquet diagram of orbit $Y = z$



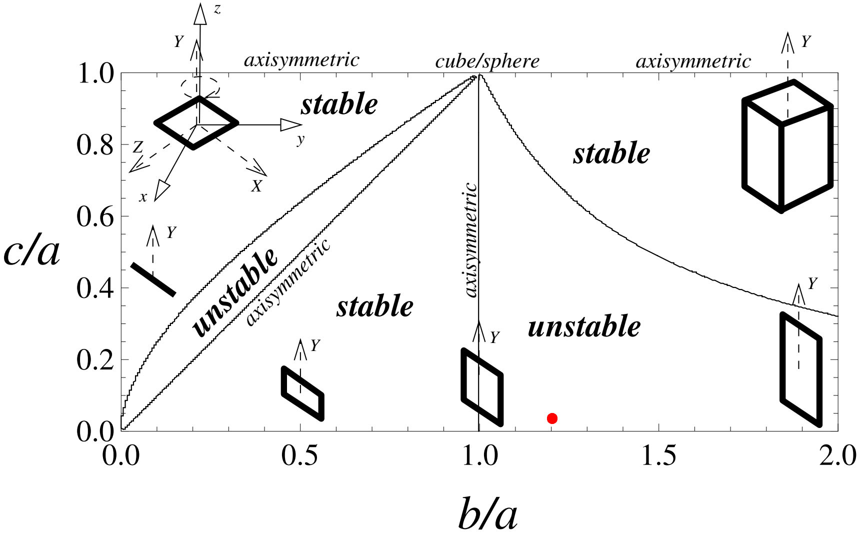
(To be compared to Yarin et al. 1997)

Result: Floquet diagram of orbit $Y = z$

$a = 1, \quad b = 1, \quad c = 0.05$

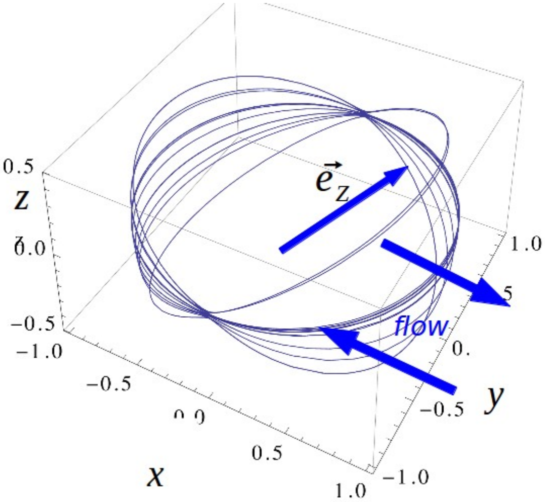


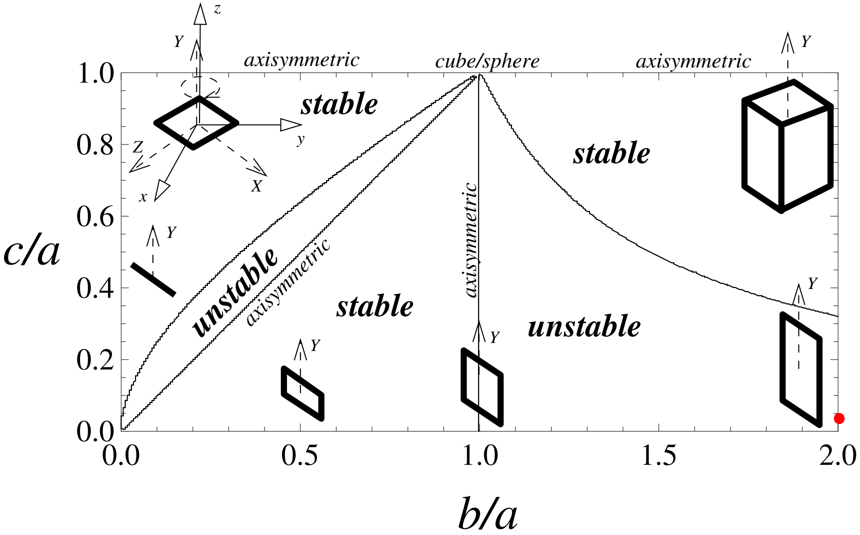
Result: Floquet diagram of orbit $Y = z$



Result: Floquet diagram of orbit $Y = z$

$a = 1, \quad b = 1.2, \quad c = 0.05$

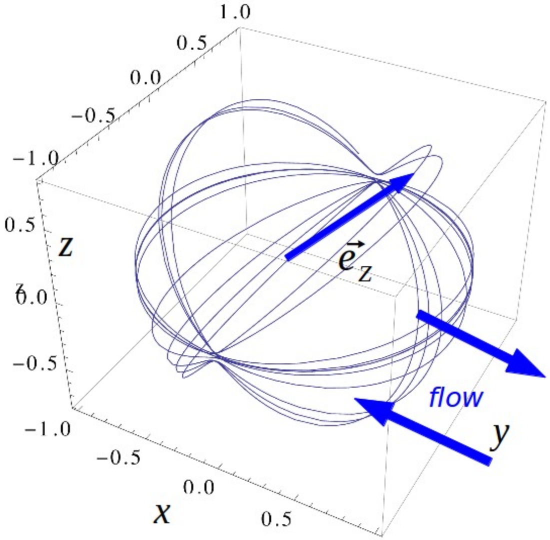




z
Y
axisymmetric
cube/sphere
axisymmetric
Y
1.0
stable
0.8
Z
y
stable
X
x
0.6
Y
c/a
axisymmetric
unstable
axisymmetric
0.4
stable
Y
0.2
Y
Y
unstable
0.0
0.0
0.5
1.0
1.5
2.0
b/a

Result: Floquet diagram of orbit $Y = z$

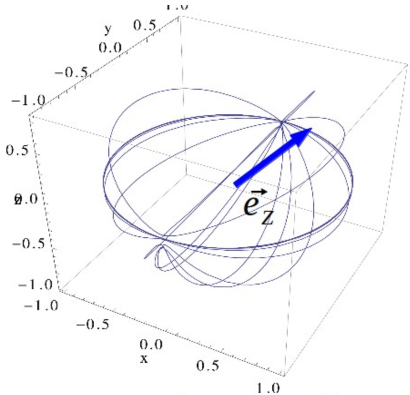
$a = 1, \quad b = 2, \quad c = 0.05$



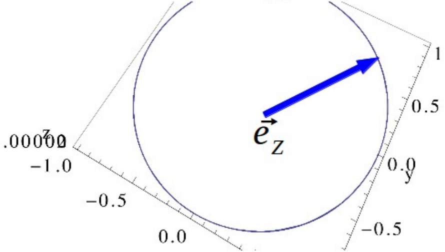
Result: Floquet diagram of orbit $Y = z$

$a = 1, \quad c = 0.05$

$b = 6$



$b = 7$



Conclusion: bi-vector formulation for triaxial objects.

$$\boxed{\dot{\vec{e}}_X = \underbrace{f_1\,(\vec{e}_Y.\mathbf{S}\vec{e}_X)\vec{e}_Y}_{\text{"}triaxial\ \ term\text{"}} + \mathbf{B}\vec{e}_X - (\vec{e}_X.\mathbf{B}\vec{e}_X)\vec{e}_X}$$

$$\boxed{\dot{\vec{e}}_Y = \underbrace{f_2\,(\vec{e}_Y.\mathbf{S}\vec{e}_X)\vec{e}_X}_{\text{"}triaxial\ \ term\text{"}} + \mathbf{C}\vec{e}_Y - (\vec{e}_Y.\mathbf{C}\vec{e}_Y)\vec{e}_Y}$$

$$\mathbf{B} = (a^2\nabla\vec{u} - c^2\nabla\vec{u}^T)/(a^2+c^2);\ f_1 = \frac{2a^2(c^2-b^2)}{(a^2+b^2)(a^2+c^2)}.$$

$$\mathbf{C} = (b^2\nabla\vec{u} - c^2\nabla\vec{u}^T)/(b^2+c^2)\ ;\ f_2 = \frac{2b^2(c^2-a^2)}{(a^2+b^2)(b^2+c^2)}.$$

- Simple formulation, cubic non-linearities only. Allows a straightforward application of Floquet theory. Low computational cost !

- **Perspectives:** use the bi-vect formulation to analyze chaotic dynamics (with Abdel Zalt, LUSAC-Cherbourg).

- Brownian objects, Fokker-Planck Eq. for $PDF(\vec{e}_X, \vec{e}_Y)$