Large-eddy simulations and Related techniques

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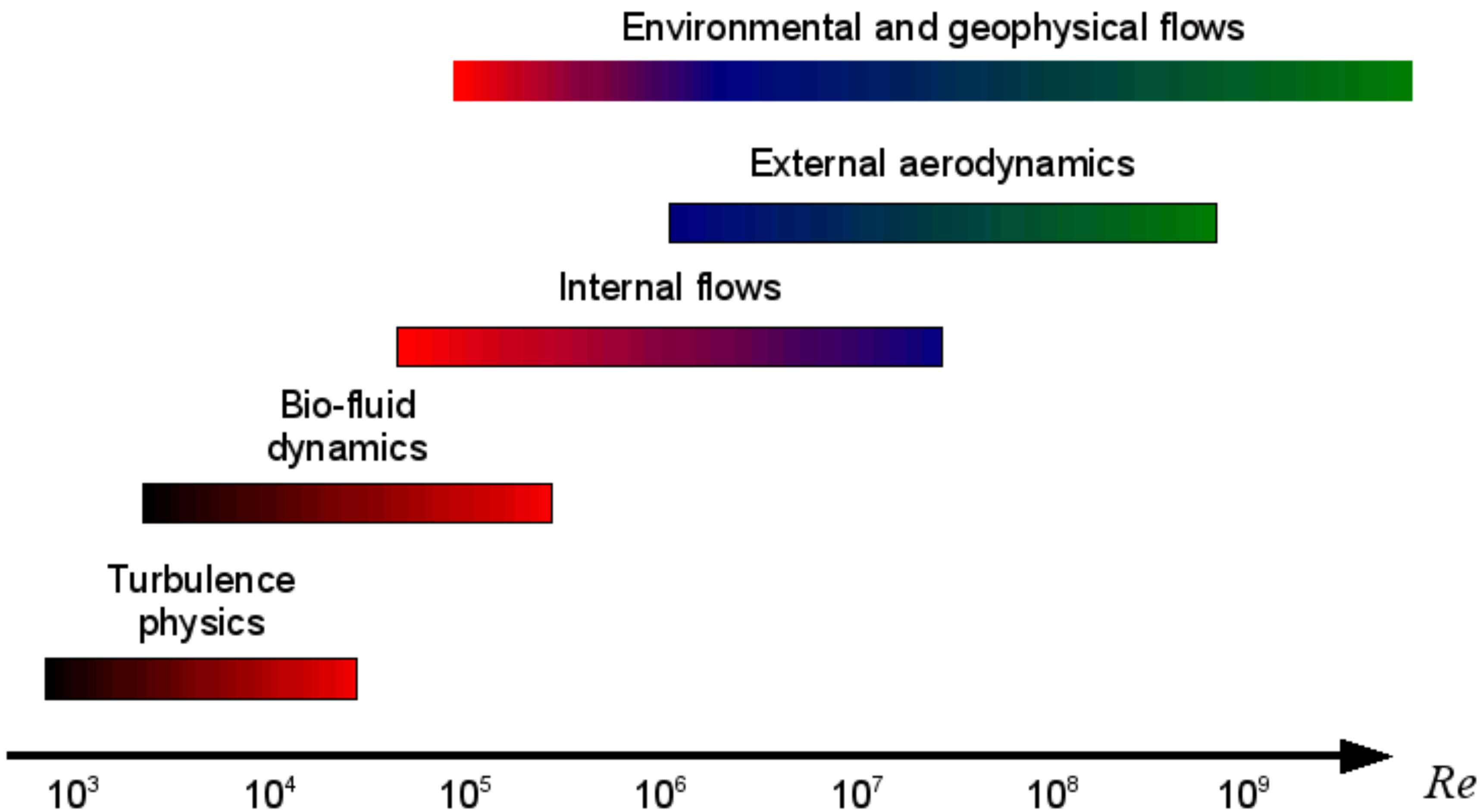
Conclusions

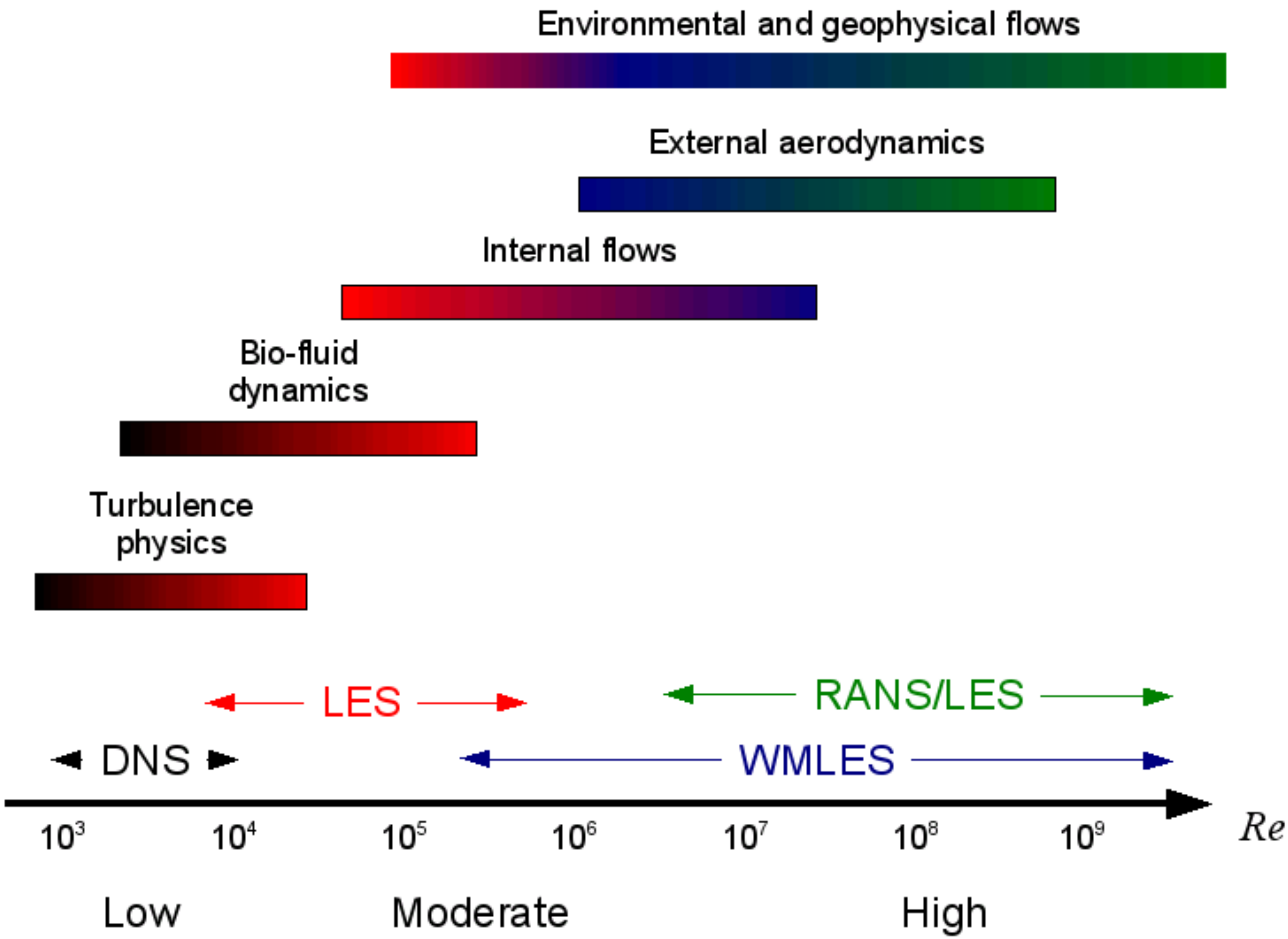
- Turbulence plays a critical role in engineering and in the physical sciences.



Conclusions

- Turbulence plays a critical role in engineering and in the physical sciences.
- Eddy-resolving techniques are becoming the most complete, general methods to solve turbulent flow problems.
  - *Include more physics*
  - *Are more accurate*
  - *Are becoming more affordable*

Roadmap



Environmental and geophysical flows
External aerodynamics
Internal flows
Bio-fluid dynamics
Turbulence physics
LES
RANS/LES
DNS
WMLES
10³
10⁴
10⁵
10⁶
10⁷
10⁸
10⁹
Re
Low
Moderate
High

Outline

- **Motivation**
- Governing equations
- Boundary conditions
- Subfilter-scale modelling
- Validation of an LES
- Applications
- Challenges
- Conclusions and course roadmap

Outline

- **Motivation:**
  - ***What is turbulence?***
  - *Review of turbulence physics*
  - *Why simulations?*
  - *Methodologies*
  - *Resolution requirements*
- Governing equations for LES
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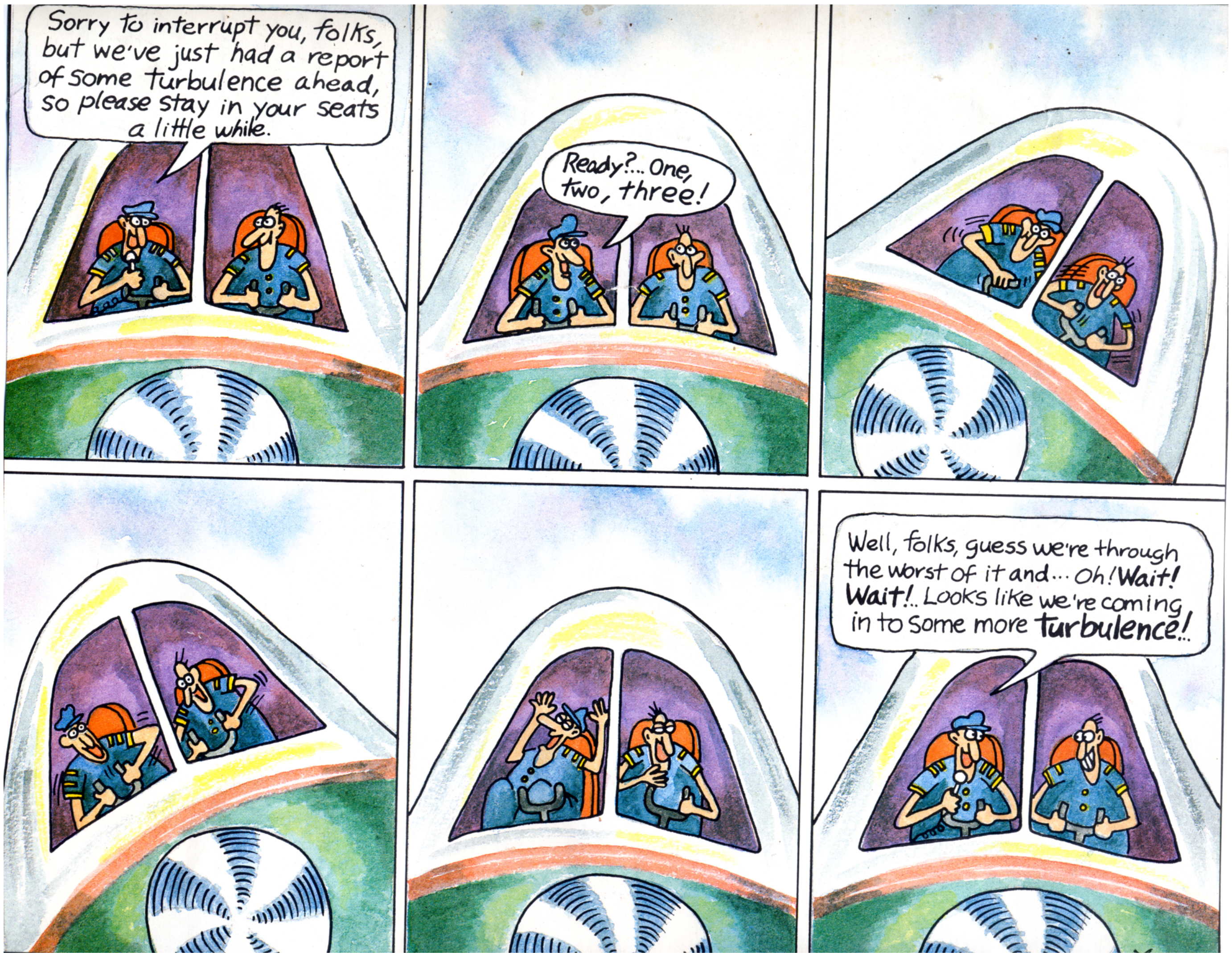
- What is turbulence?



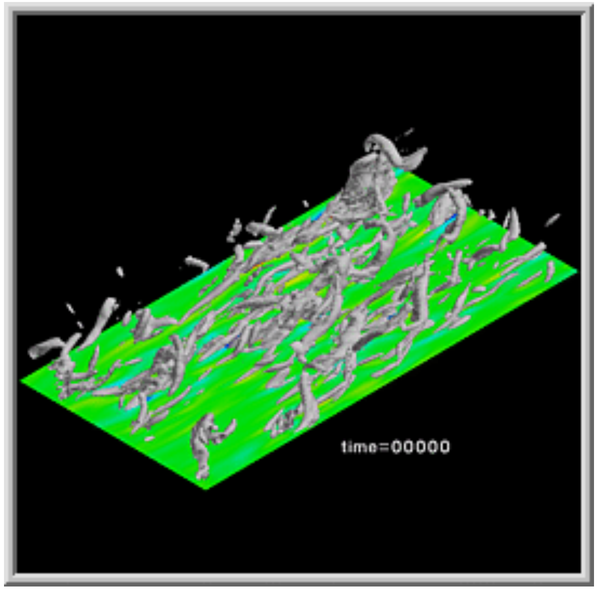
Motivation

- Turbulence is the chaotic state of fluid motion that occurs when convective effects are much larger than viscous effects.
  - *Ship:* $Re = o(10^9)$
  - *Ocean:* $Re = o(10^9)$
  - *Aircraft:* $Re = o(10^8)$
  - *Car:* $Re = o(10^6)$
  - *Golf ball:* $Re = o(10^5)$
  - *Artery:* $Re = o(10^3)$

Motivation

- Turbulence is the chaotic state of fluid motion that occurs when convective effects are much larger than viscous effects.
- Turbulence contains vorticity: $\omega = \nabla \times \mathbf{V}$
  - *Coherent vortical motions* $\Rightarrow$ *Eddies*

- Turbulence is the chaotic state of fluid motion that occurs when convective effects are much larger than viscous effects.
- Turbulence contains vorticity: $\omega = \nabla\times\mathbf{V}$
  - *Coherent vortical motions ⇒ Eddies*
  - *Deterministic but occur at random locations*
  - *Turbulence is not a completely stochastic phenomenon*
  - *Statistical descriptions are inadequate.*



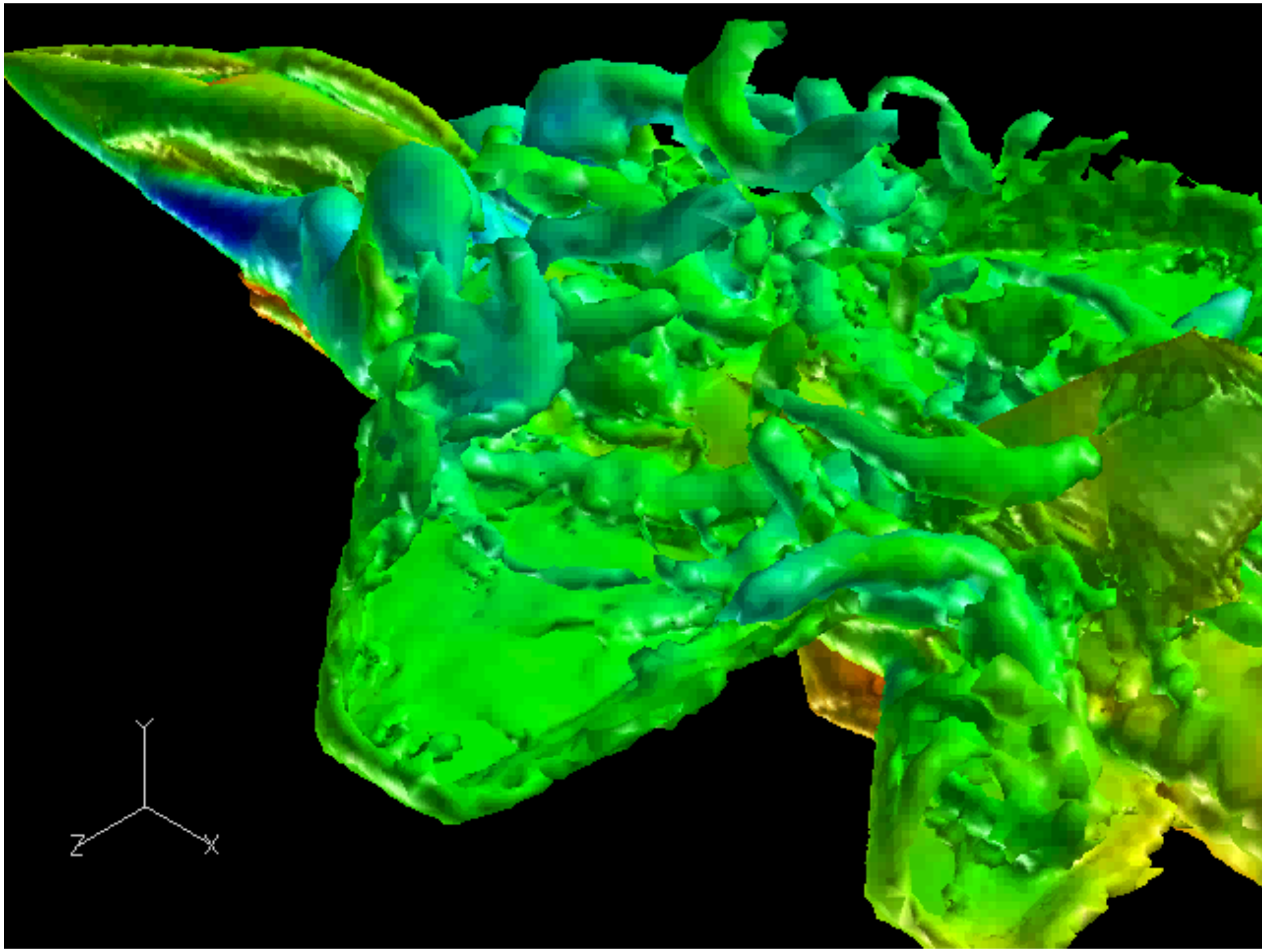
Coherent eddies in turbulent channel flow (Bewley, Temam and Moin 2000).

Motivation

- Turbulence is the chaotic state of fluid motion that occurs when convective effects are much larger than viscous effects.
- Turbulence contains vorticity: $\omega = \nabla \times \mathbf{V}$
- Turbulence is not a completely random phenomenon.
- Turbulence enhances mixing.
- Turbulence is the natural state of fluid motion.

Motivation

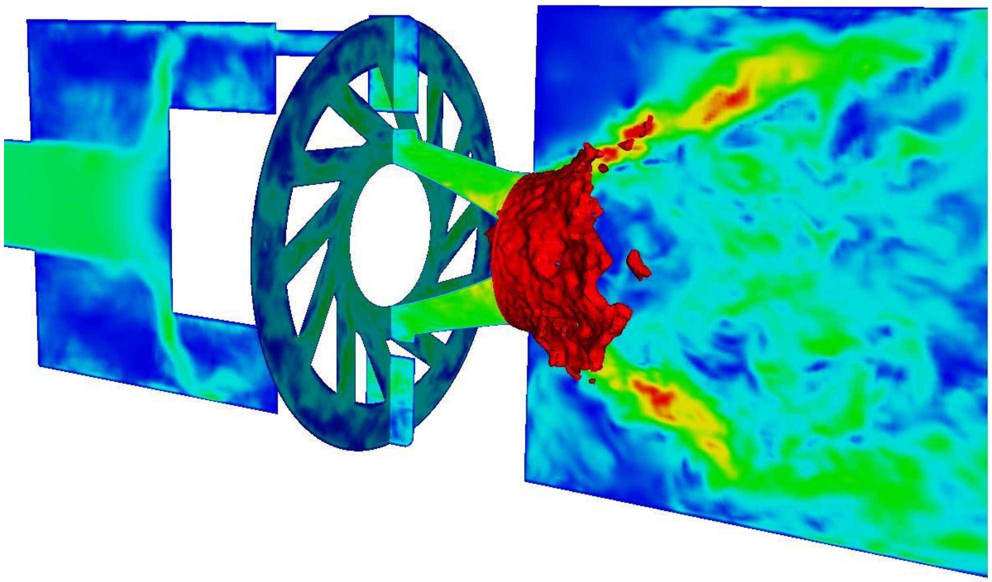
- Turbulence is all around us
  - *Engineering devices*
    - Aerospace applications



Flow over an F-16
at 45^o angle of attack

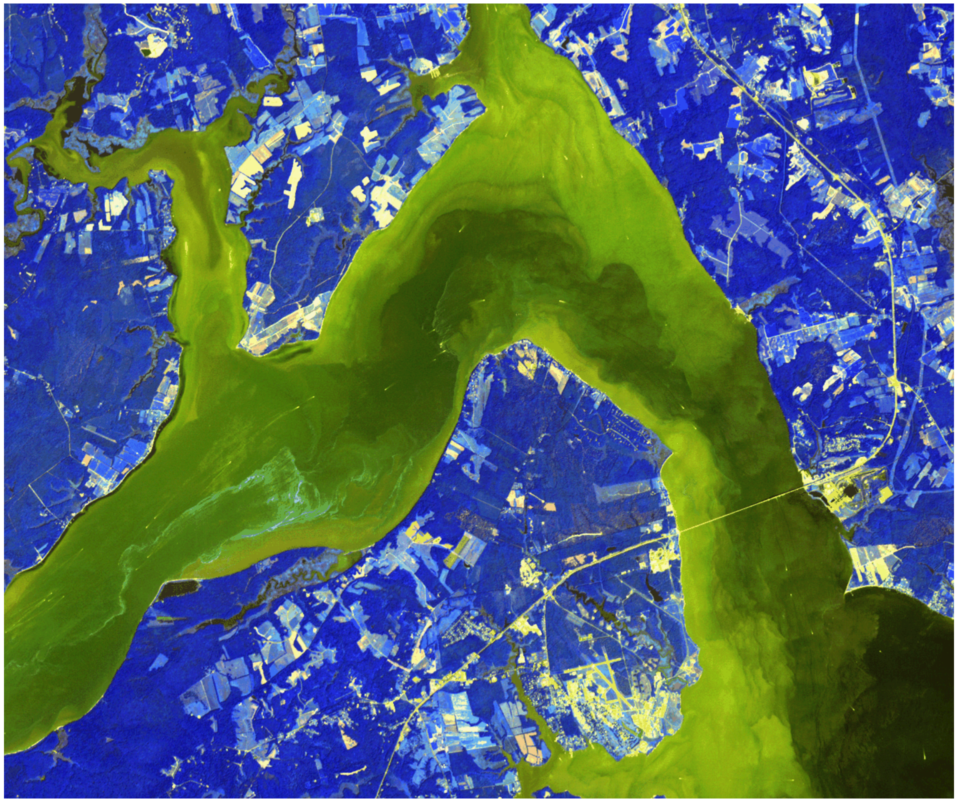
Calculation by
J. Forsythe
(Cobalt)

- Turbulence is all around us
  - *Engineering devices*
    - Aerospace applications
    - Naval applications
    - Vehicle aerodynamics
    - Combustion systems



Flow in a combustor
Calculation by H. Pitsch
(Stanford University)

- Turbulence is all around us
  - *Engineering devices*
    - Aerospace applications
    - Naval applications
    - Vehicle aerodynamics
    - Combustion systems
  - *Geophysical applications*
    - Oceanography



Frontal Features and Vortices in the Tidal Potomac

Motivation

- Turbulence is all around us
  - *Engineering devices*
    - Aerospace applications
    - Naval applications
    - Vehicle aerodynamics
    - Combustion systems
  - *Geophysical applications*
    - Oceanography
    - Meteorology and weather prediction
    - Environmental engineering
  - *Biological flows*

Motivation

- Turbulence is all around us
  - *Engineering devices*
    - Aerospace applications
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    - Vehicle aerodynamics
    - Combustion systems
  - *Geophysical applications*
    - Oceanography
    - Meteorology and weather prediction
    - Environmental engineering
  - *Biological flows*

$\Rightarrow$ It is critical to predict and analyse the effects of turbulence on mass, momentum and energy transport

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The Energy Cascade

- Consider a high-Reynolds-number turbulent flow.
  - *Length scale* L
  - *Velocity scale* U
  - *Reynolds number* UL/ν

- Turbulence made up of eddies of different sizes:
  - *Length-scale* ℓ
  - *Velocity scale* $u(\ell)$
  - *Time scale* $\tau(\ell)$

The Energy Cascade

- Largest eddies: $\ell_o \sim L; \quad u_o \sim U$

  $$\Rightarrow \; Re_o = \frac{u_o \ell_o}{\nu} = o(Re) >> 1$$

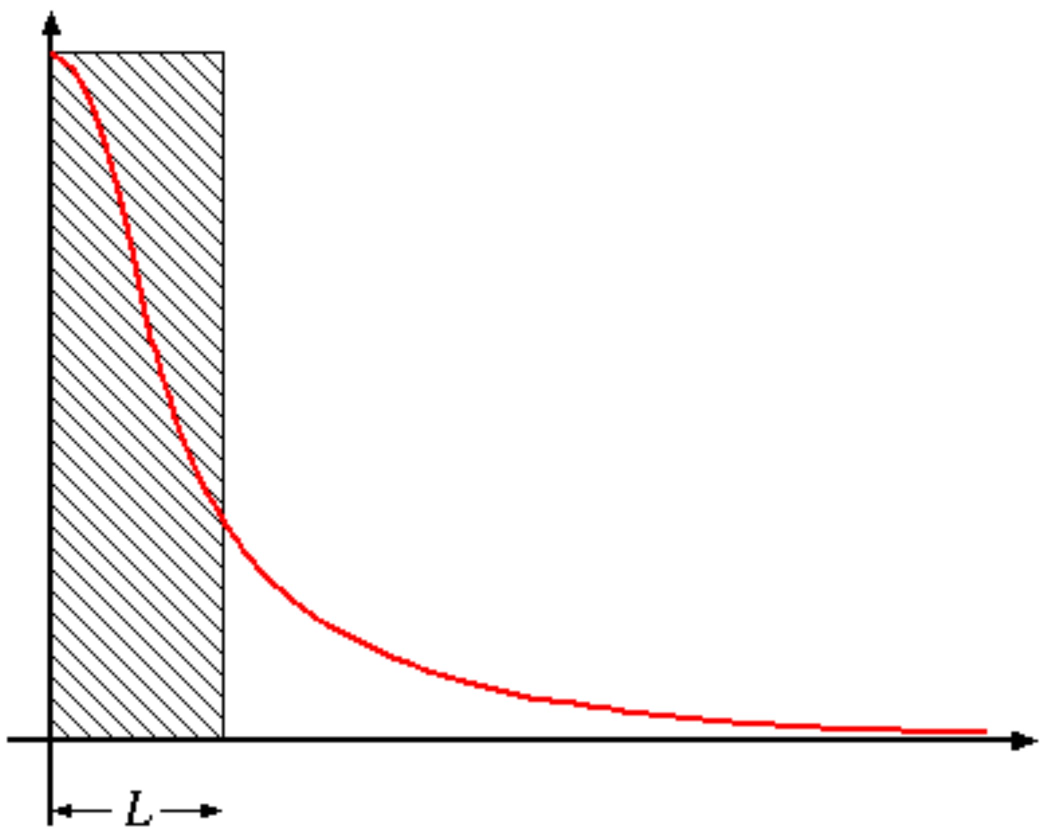
  ⇒ Dissipation is unimportant

- Large eddies break up and transfer their energy to smaller eddies.
- The process continues until the smallest eddies, whose Re is so small that dissipation becomes important.

The Integral Scale

- Typical length scale of the largest eddies is the integral scale L

$$L_{ij} = \int_0^\infty R_{ij}(r)dr$$



The Energy Cascade

- Energy
  - *Enters at the large scales (production likes anisotropy)*
  - *Is progressively transferred to smaller scales*
  - *Is dissipated by the smallest scales*
- Dissipation
  - *Takes place at the smallest scales.*
  - *Is determined by the production (largest scales).*
  - *Energy* $\propto u_o^2$ *Time-scale* $\propto \ell_o / u_o$

$$\Rightarrow \varepsilon \propto \frac{u_o^3}{\ell_o}$$

(Independent of ν)

Local Isotropy

- The large eddies are affected by the b.c.
  $\Rightarrow$ are anisotropic
- As they break up they lose memory of the b.c.s
  $\Rightarrow$ become more isotropic.

- Kolmogorov's hypothesis of local isotropy:
  - *At sufficiently high Reynolds number, the small-scale motions* $\ell << \ell_o$ *are statistically isotropic*

The Kolmogorov Scales

- At small scales, ν is important, ε determines the rate of energy transfer.
- Kolmogorov's 1st similarity hypothesis:
- At sufficiently high Reynolds number, the statistics of the small-scale motions are uniquely determined by ε and ν.
- We can build length-, time- and velocity scales using ε and ν:

Kolmogorov scales

$$\begin{aligned} \eta &= \left(\nu^3/\varepsilon\right)^{1/4} \\ u_\eta &= \left(\nu\varepsilon\right)^{1/4} \\ \tau_\eta &= \left(\nu/\varepsilon\right)^{1/2} \end{aligned}$$

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Kolmogorov scales

$$\begin{aligned} \eta/\ell_o &\sim Re^{-3/4} \\ u_\eta/u_o &\sim Re^{-1/4} \\ \tau_\eta/\tau_o &\sim Re^{-1/2} \end{aligned}$$

The Inertial Subrange

- Intermediate range: $\eta << \ell << \ell_o$
- Re >>1
  $\Rightarrow$ Viscous forces << inertia forces
  $\Rightarrow$ ν is unimportant.
- Kolmogorov's 2nd similarity hypothesis:
- At sufficiently high Reynolds number, the statistics of the motions of scale $\eta << \ell << \ell_o$ are uniquely determined by ε independent of ν .

$$\Downarrow$$

Inertial (sub)range of turbulence

The Inertial Subrange

- In the inertial subrange:

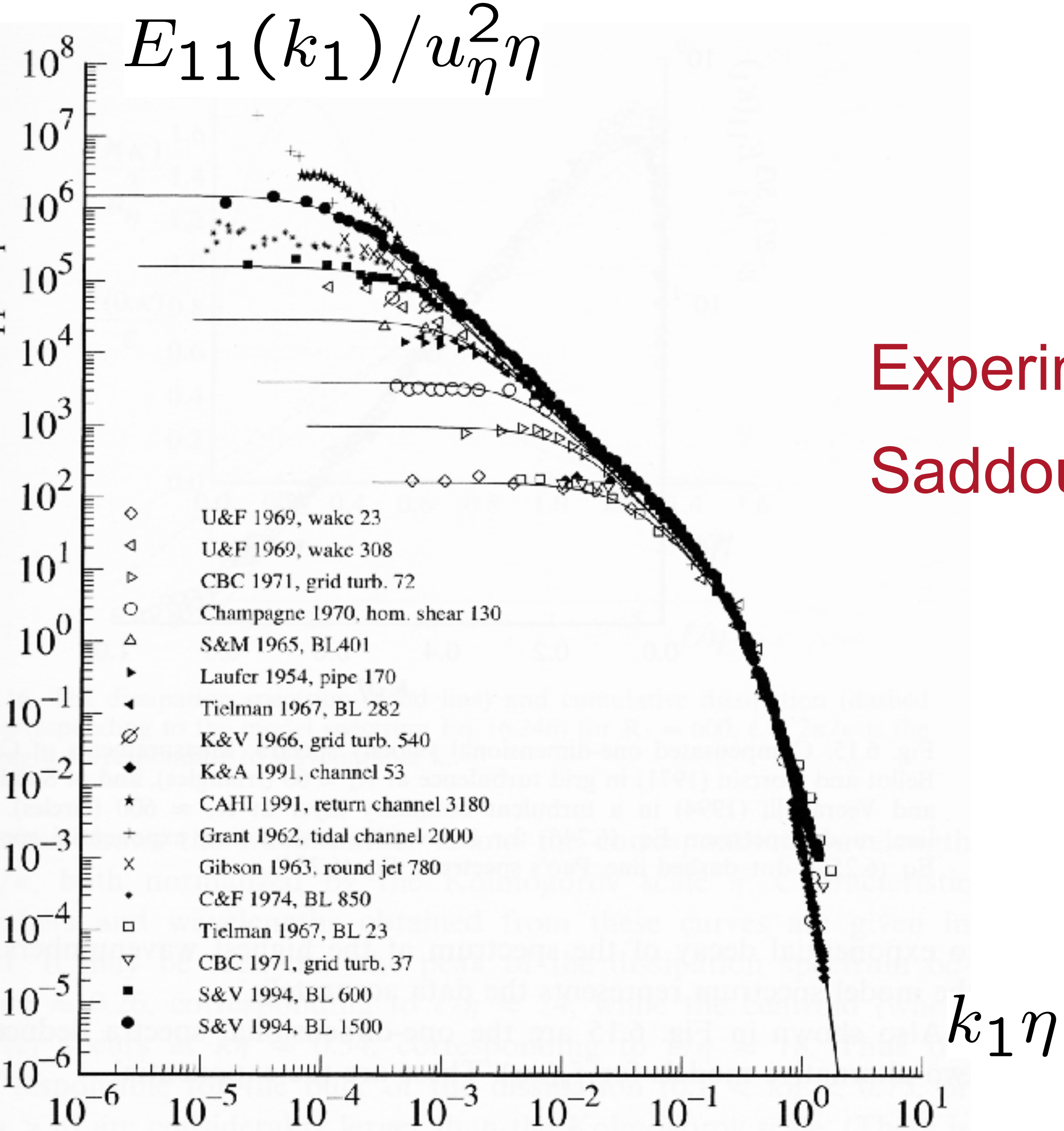
$$2 \times \text{TKE} = \int_0^\infty E(\kappa) d\kappa; \quad E(\kappa) = f(\kappa, \varepsilon)$$

$$\Rightarrow E(\kappa) = C_K \varepsilon^{2/3} \kappa^{-5/3}$$

- C_κ is the Kolmogorov constant

Kolmogorov spectrum

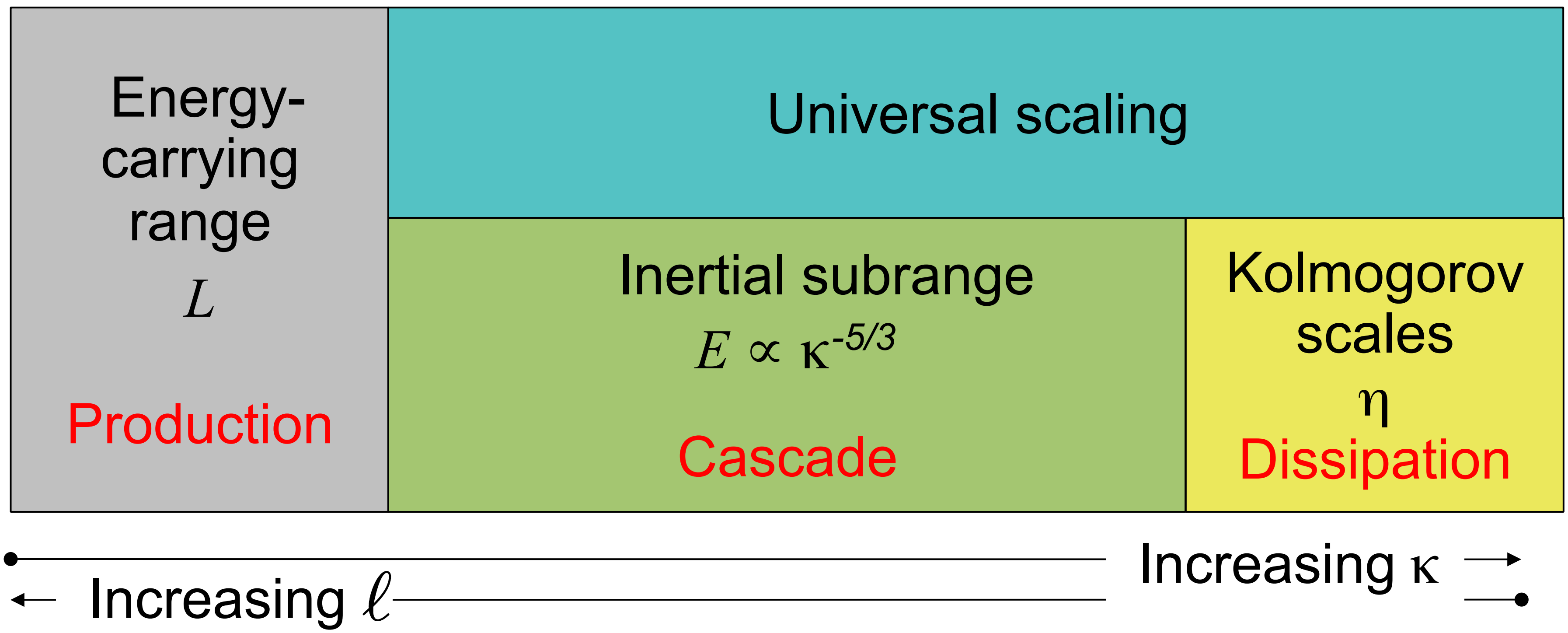
Universal Scaling



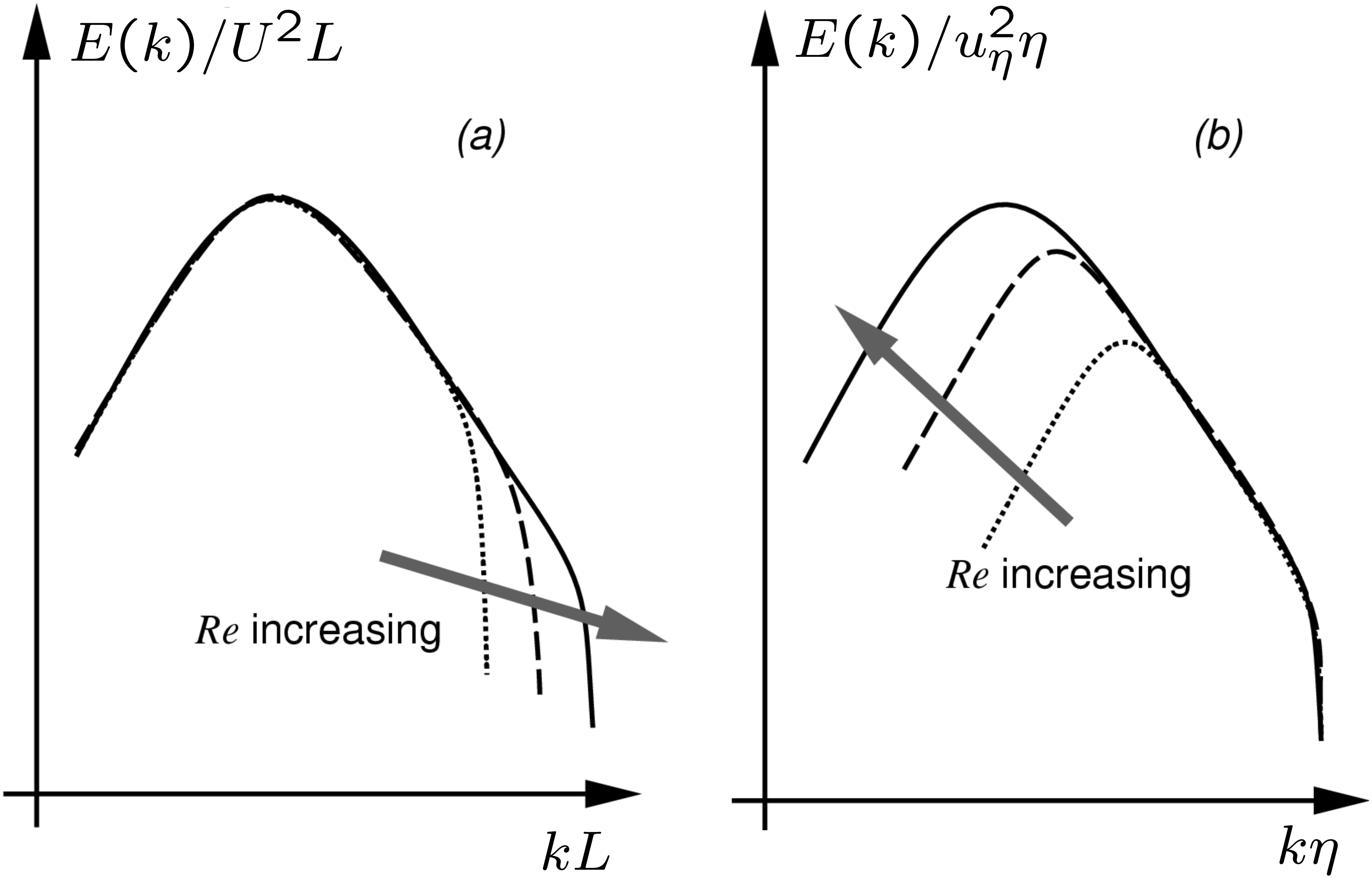
Experimental verification:

Saddougui & Veeravalli (1994)

The Energy Cascade



Reynolds Number Effects



Free-Shear Flows

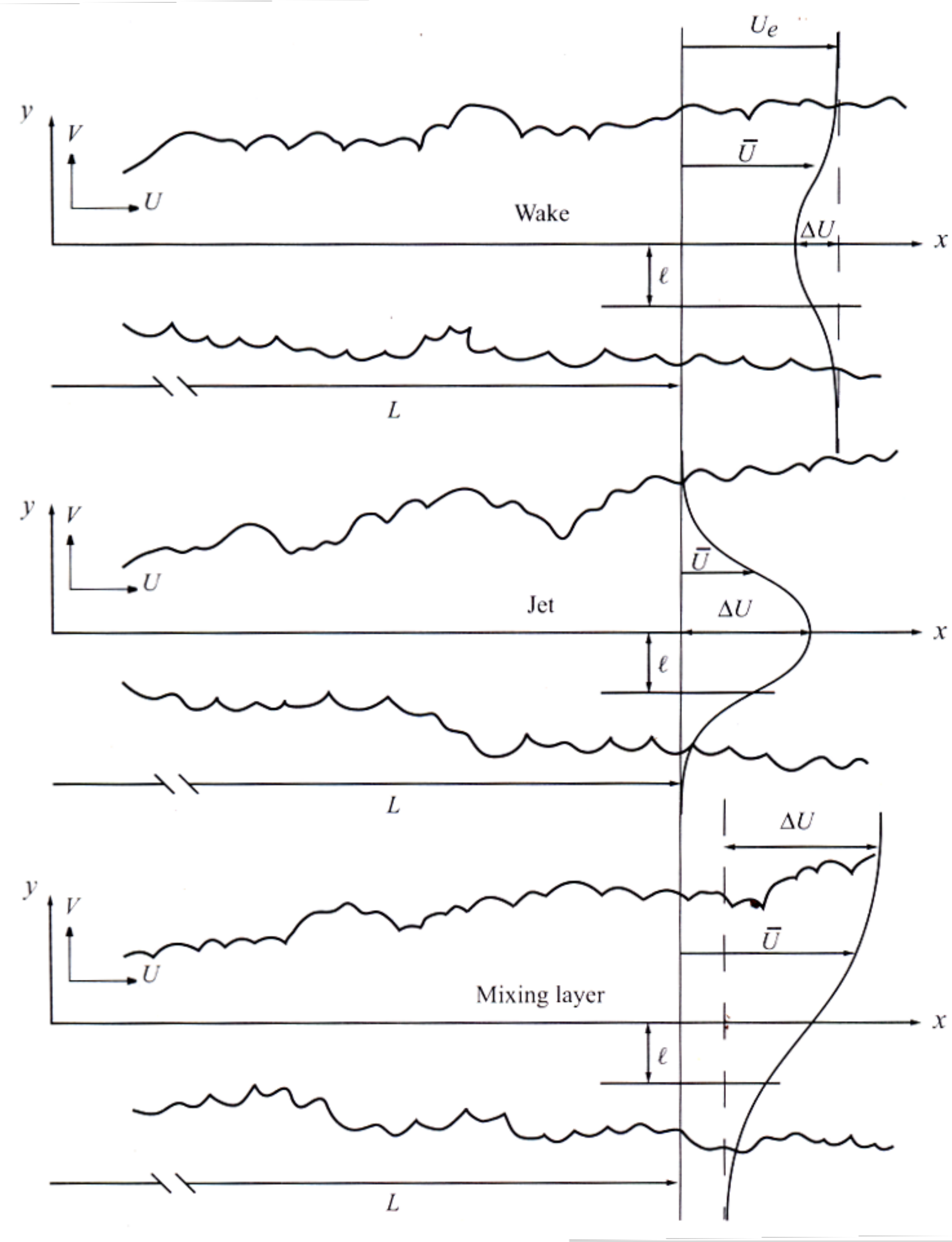
- Flows with non-zero mean-velocity gradient, and far from solid boundaries.
- Slow streamwise development:

$$\frac{\partial}{\partial x} << \frac{\partial}{\partial y}, \frac{\partial}{\partial r}$$

- Self-similarity occurs: a length-scale ℓ_r and a velocity-scale U_r can be found such that the normalized statistics depend on $\eta = y/\ell_r$ and not on x and y.

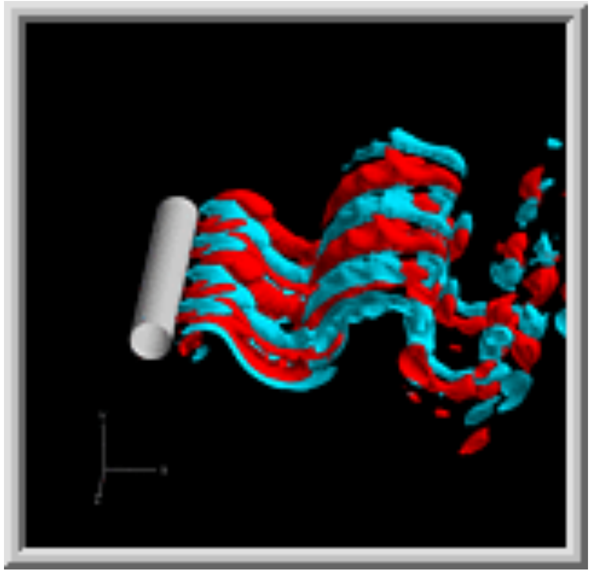
Free-Shear Flows

- Three typical cases:
  - *Wake*
  - *Jet*
  - *Mixing layer*



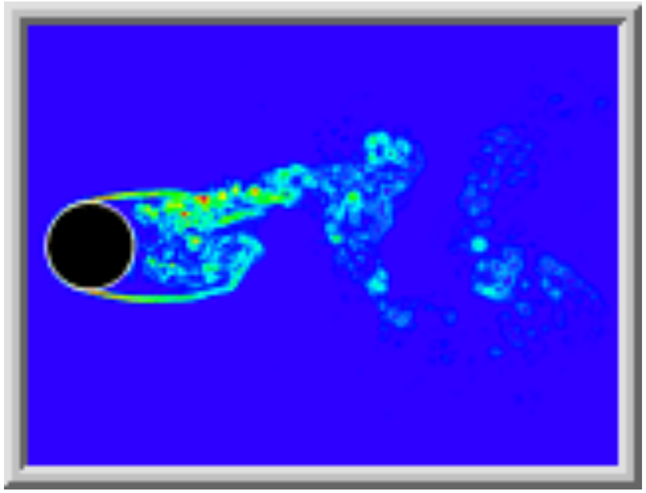
Wakes

- Most studied cases:
  - *Cylinder*
  - *Sphere*
- Low Reynolds number:
  - *Vortex shedding (Kármán street)*
  - *Three-dimensionality (rib vortices)*
- Laminar boundary layer:
  - *Subcritical*
  - *Early separation*
  - *Transition takes place in the shear layer*
  - *Well-defined vortex shedding*
- Turbulent boundary layer
  - *Supercritical*



Wakes

- Most studied cases:
  - *Cylinder*
  - *Sphere*
- Low Reynolds number:
- Laminar boundary layer:
  - *Subcritical*
  - *Early separation*
  - *Transition takes place in the shear layer*
  - *Fairly well-defined vortex shedding*
- Turbulent boundary layer



Wakes

- Most studied cases:
  - *Cylinder*
  - *Sphere*
- Low Reynolds number
- Laminar boundary layer
- Turbulent boundary layer
  - *Coherent eddies are less evident but still exist*
  - *Delayed separation on body gives lower drag.*

- Self-similarity exists far downstream (80-150 D) with

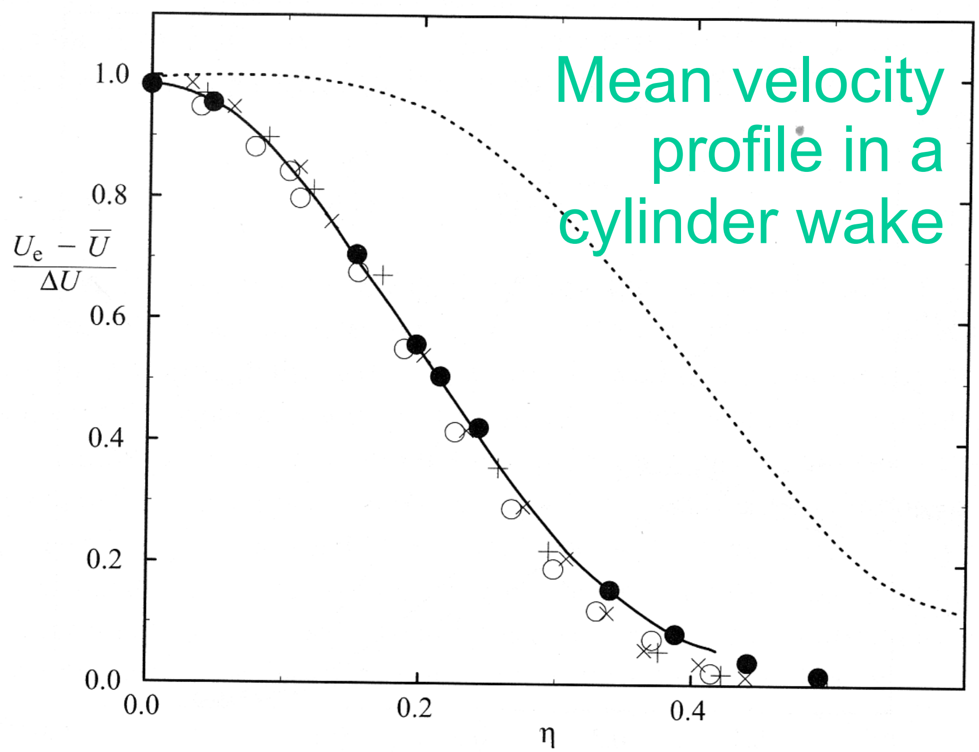
$$U_r = \Delta U = U_\infty - U_c$$
$$\ell_r = \text{Distance where } U = U_c + \Delta U/2$$

- Spreading rate for similarity:

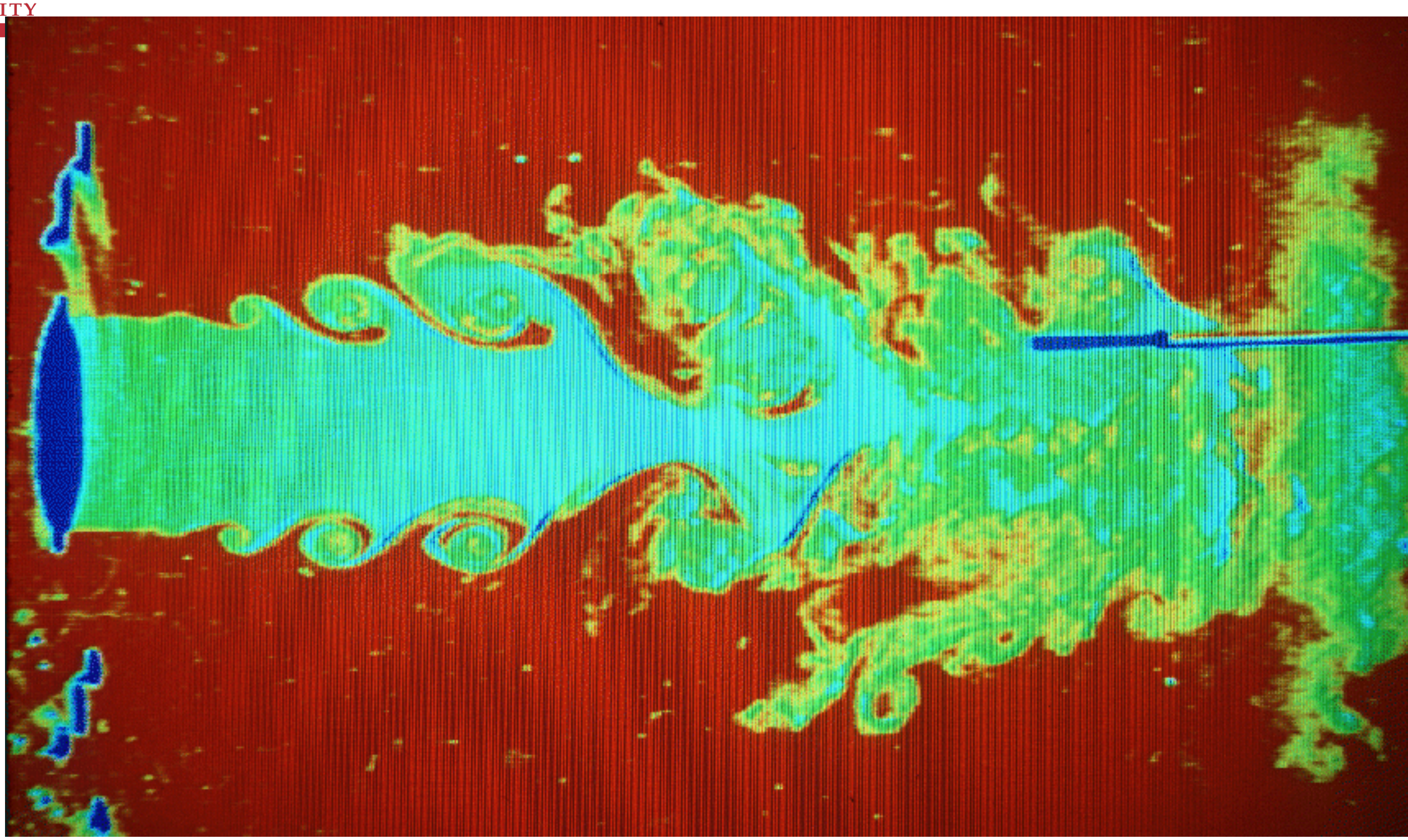
$$\ell \sim x^{1/2}$$

- Velocity deficit decay rate for similarity:

$$\Delta U \sim x^{-1/2}$$



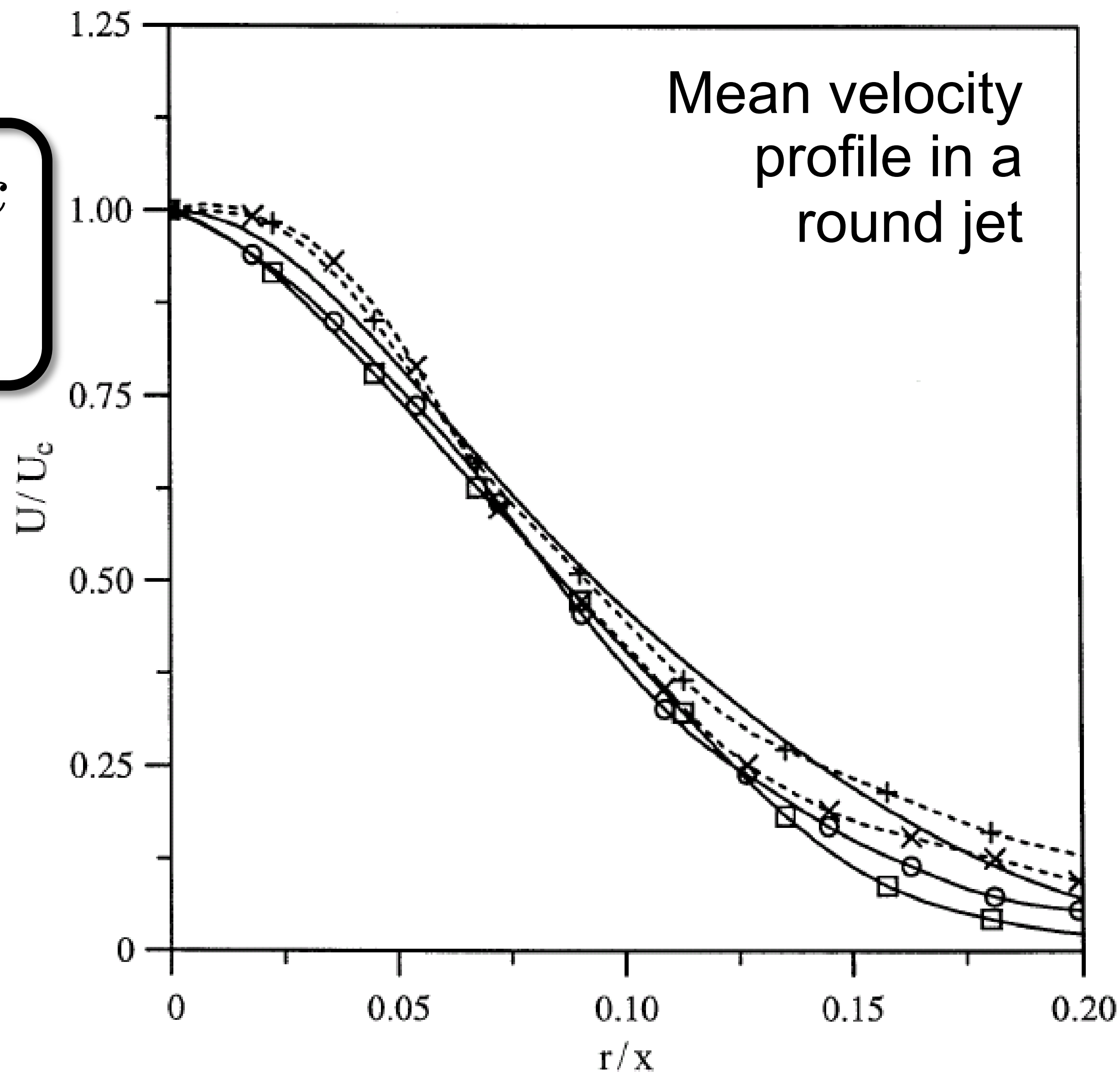
Round Jet



- The jet has excess momentum and spreads outwards.
- Instabilities in the near-jet result in vortex rings.
- At high *Re* the rings rapidly break down.

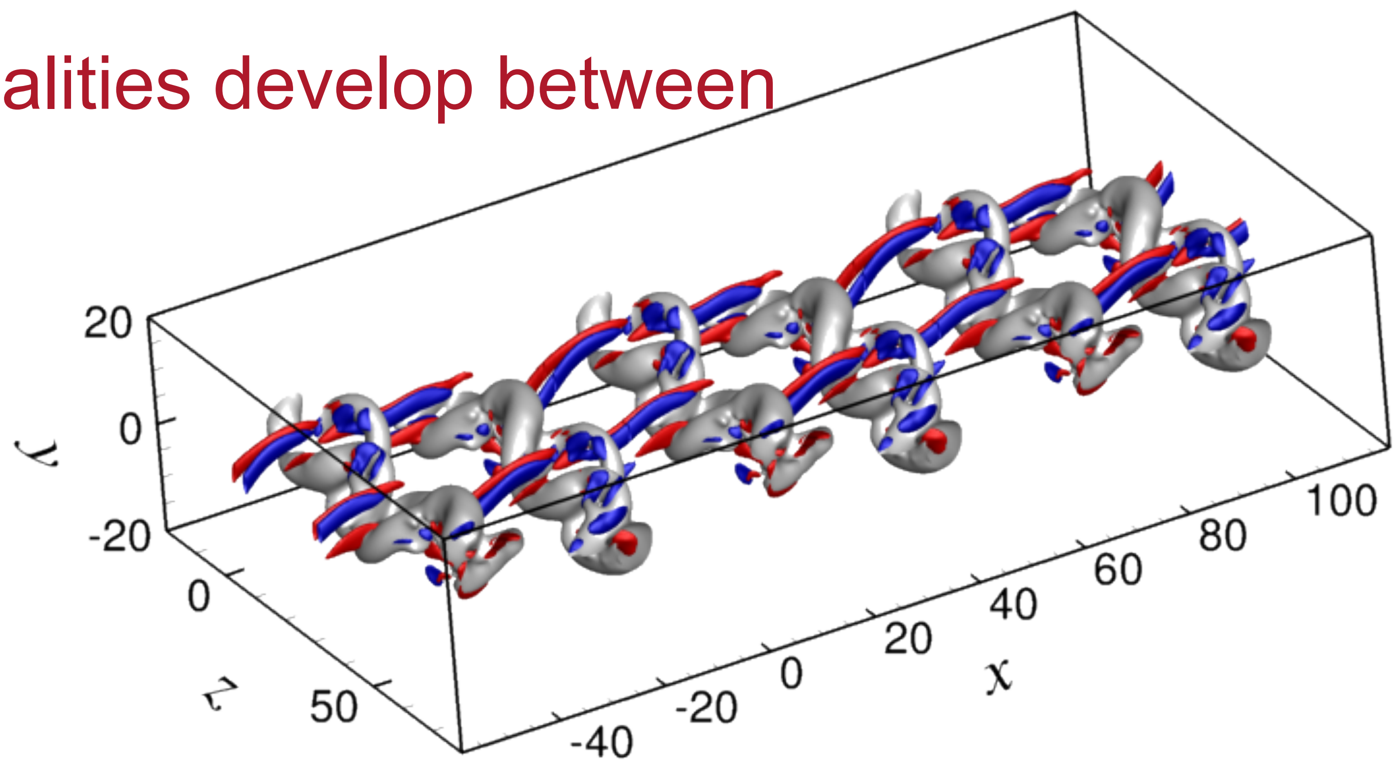
- Similarity with

$$\ell_r = \text{halfwidth} \sim x$$
$$U_r = U_c \sim x^{-1}$$



- The inflectional instability results in the formation of spanwise rollers that pair and grow.

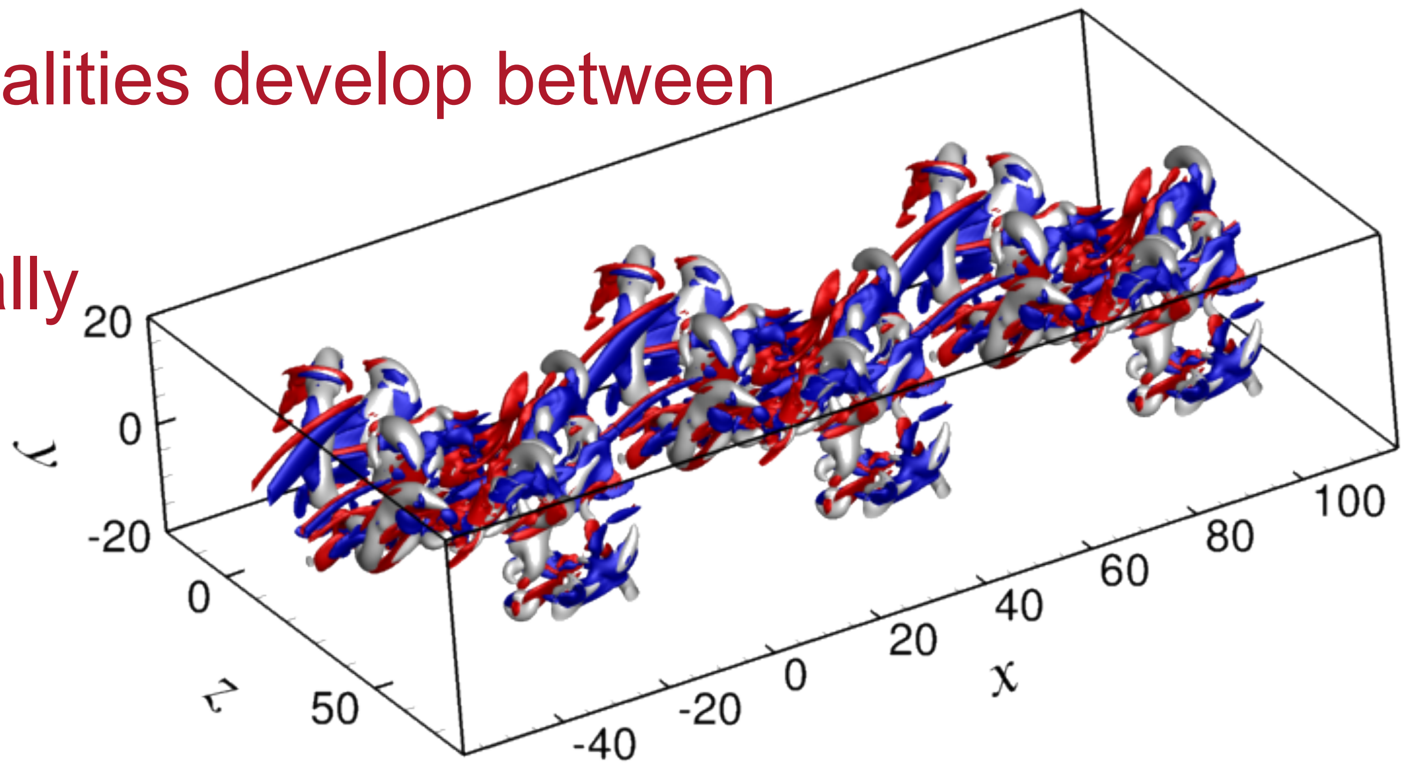
- Three-dimensionalities develop between the rollers



Mixing Layer

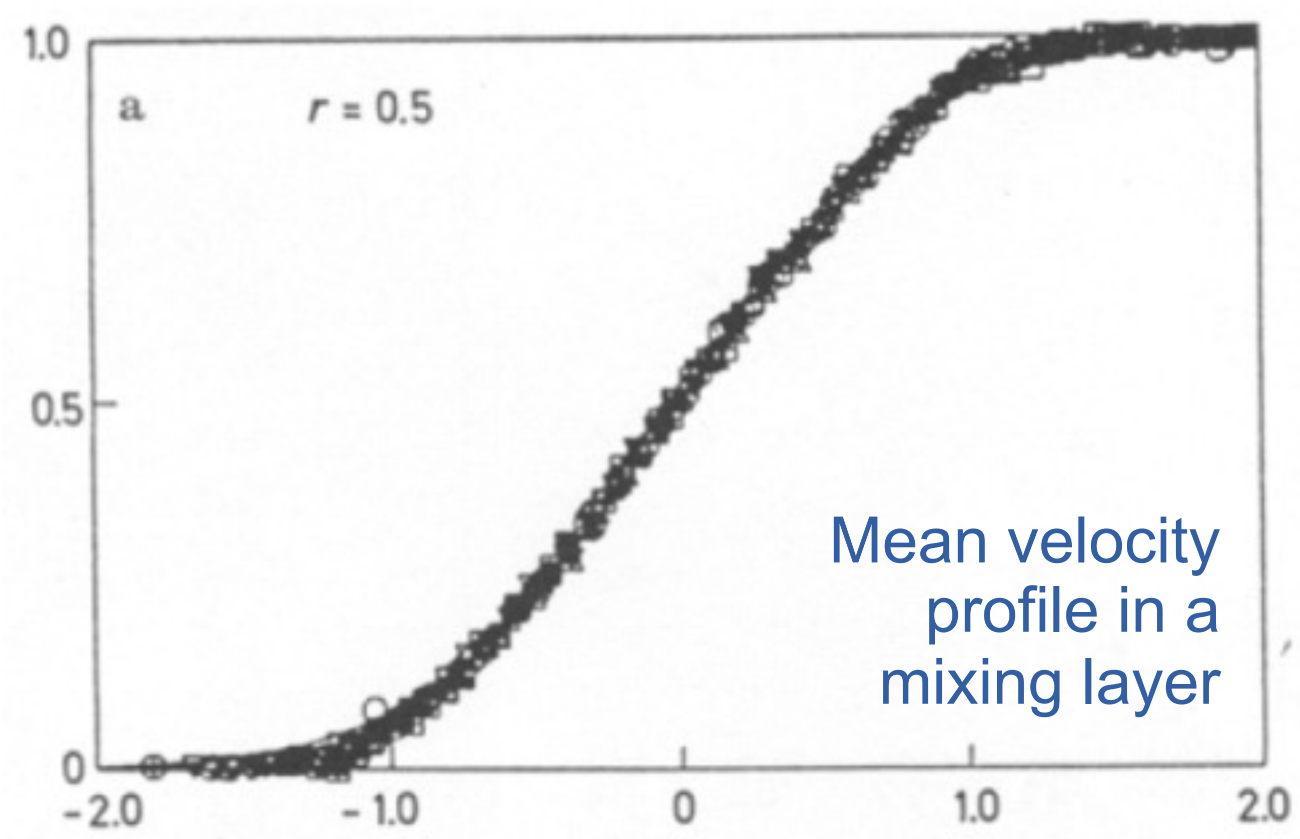
- The inflectional instability results in the formation of spanwise rollers that pair and grow.

- Three-dimensionalities develop between the rollers
- The flow eventually transition to turbulence



Mixing Layer

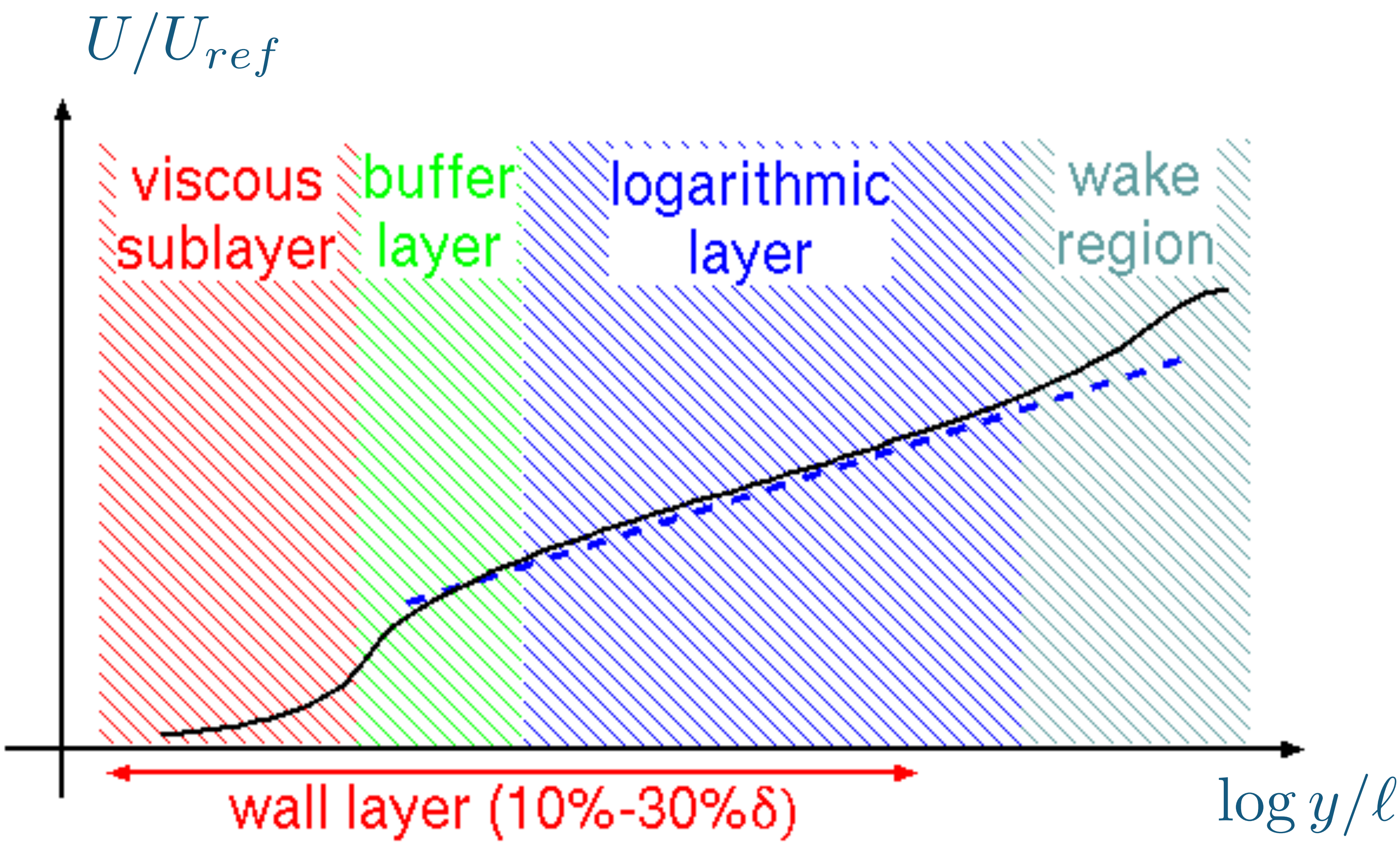
- Self-similarity is observed with $\ell_r \sim x$



Wall-Bounded Flows

- Very common in engineering applications.
  - *Pipes and ducts.*
  - *External aerodynamics.*
  - *Meteorology.*
  - *Internal combustion engines.*

- Four regions of the flow, different physics



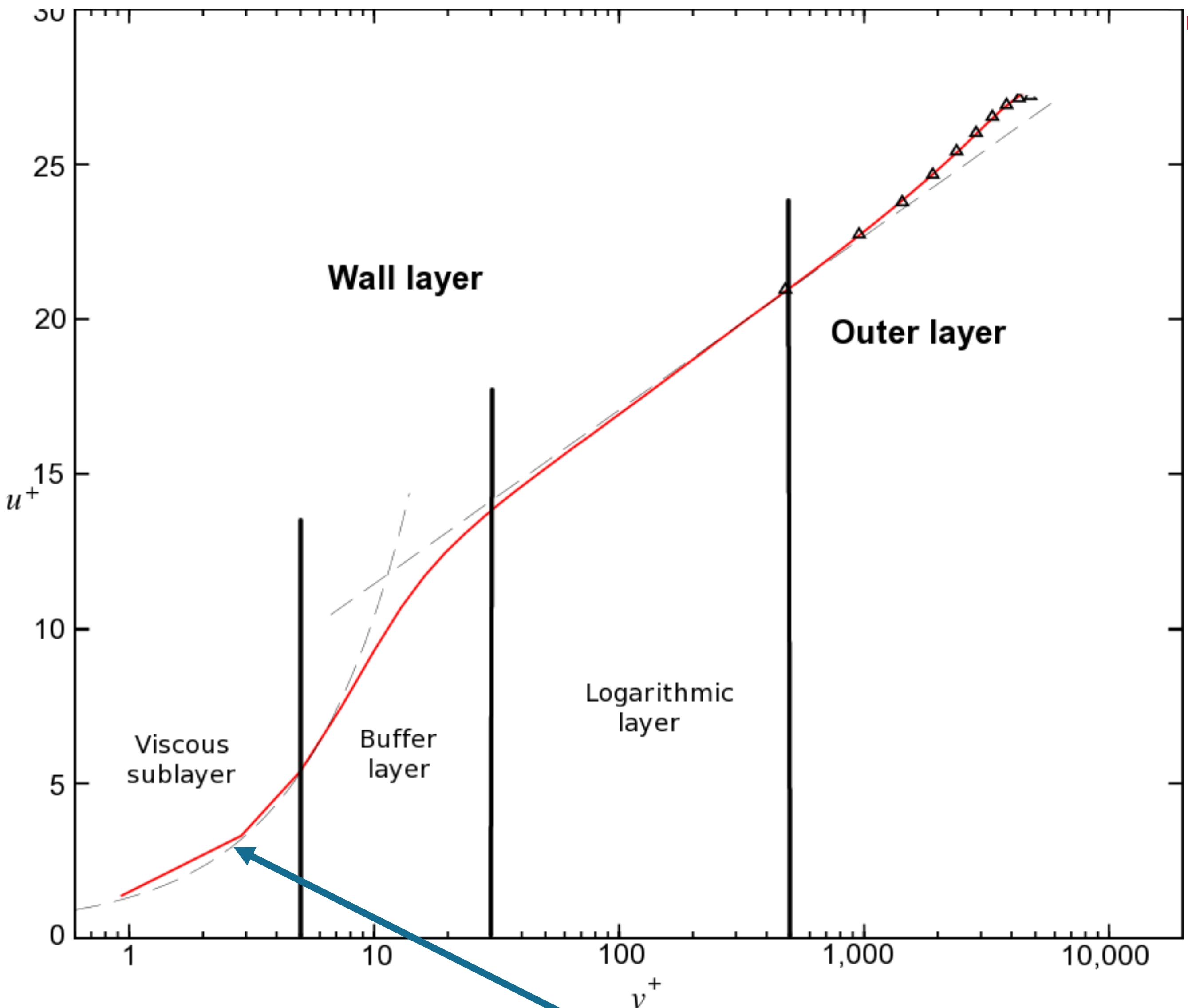
Viscous Sublayer

- Reynolds shear stress $-\langle u'v' \rangle \propto y^3$ is small.
- τ is due to viscous contribution only.
- Length and velocity scales should be obtained using only the wall stress τ_w, ρ and ν.

$$u_\tau = \left(\frac{\tau_w}{\rho}\right)^{1/2} \quad \text{and} \quad y_f = \frac{\mu}{(\rho \tau_w)^{1/2}} = \frac{\nu}{u_\tau}$$

$$\boxed{u^+ = y^+}$$

Viscous Sublayer



Linear behaviour
for y^+<5

Logarithmic Layer

- Reynolds stresses >> viscous stresses.
- The flow is unaffected by the outer flow.
- All turbulence should depend on τ_w and distance from the wall:

$$\frac{dU}{dy} = \frac{u_\tau}{\kappa y}$$

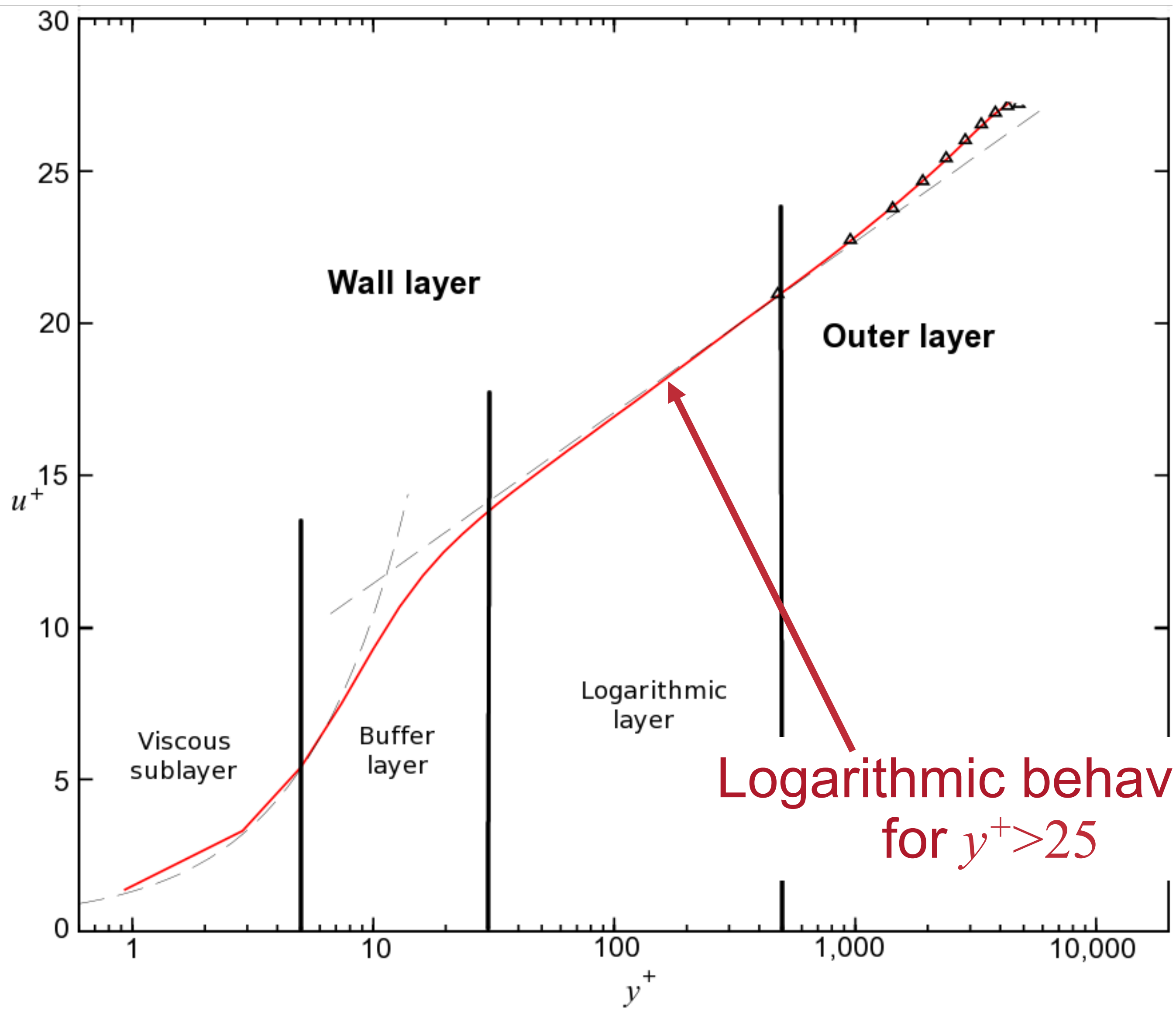
$$-\langle u'v' \rangle = \tau_w$$

κ is a constant (von Kármán constant).

- Integrating...

$$\boxed{u^+ = \frac{1}{\kappa} \log y^+ + B}$$

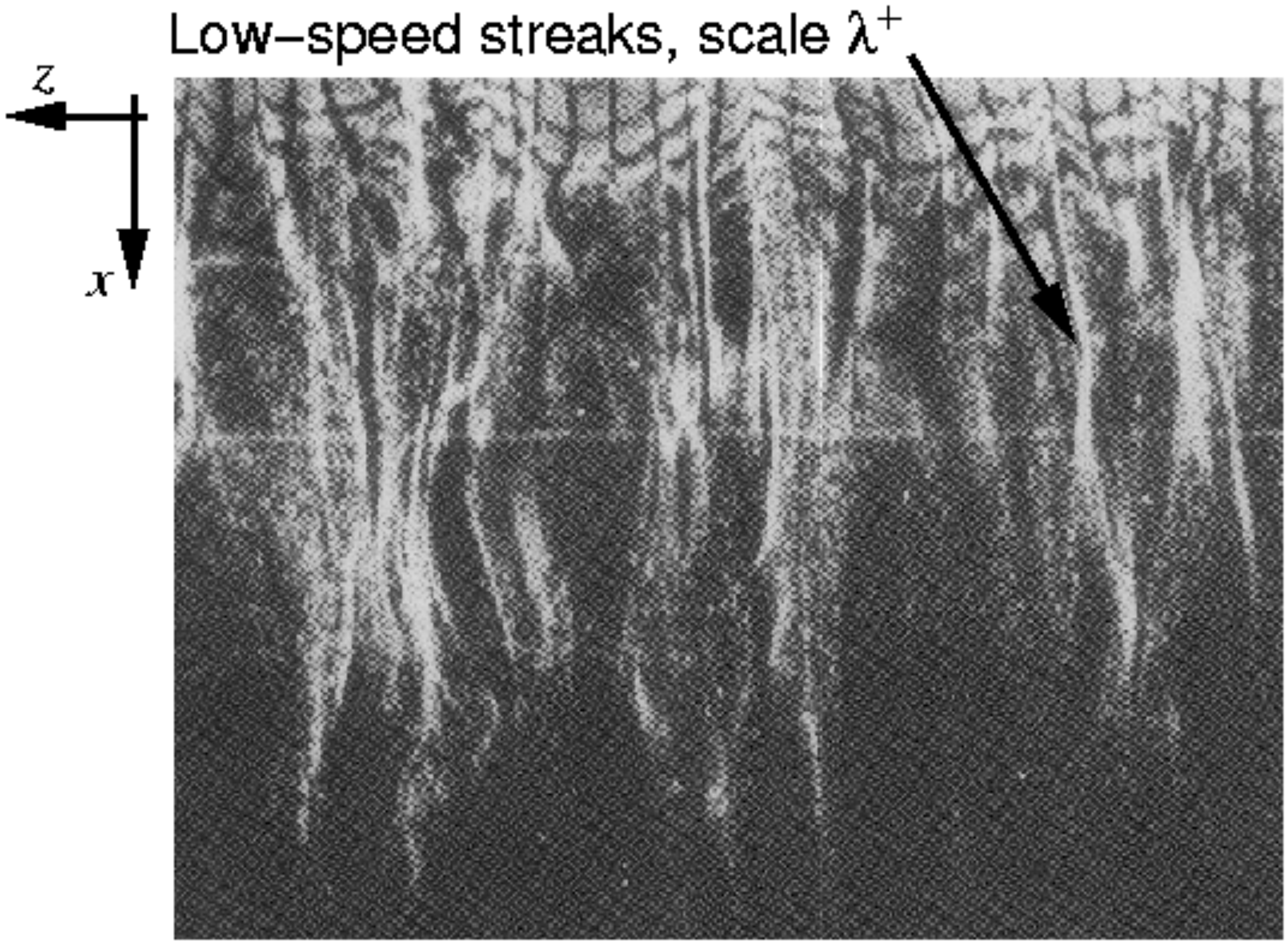
Logarithmic Layer

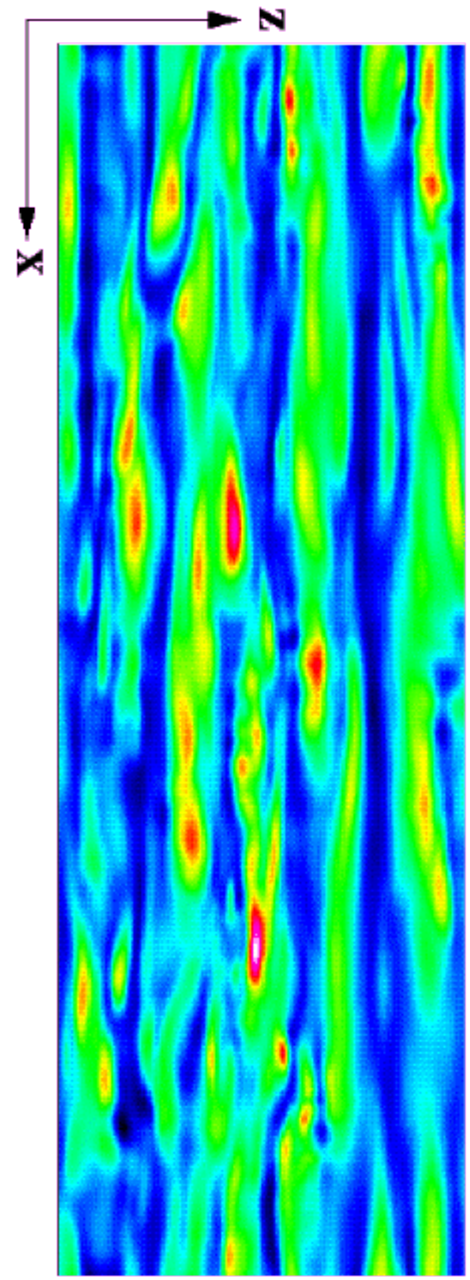


Logarithmic behaviour for $y^+ > 25$

Wall-Layer Eddies

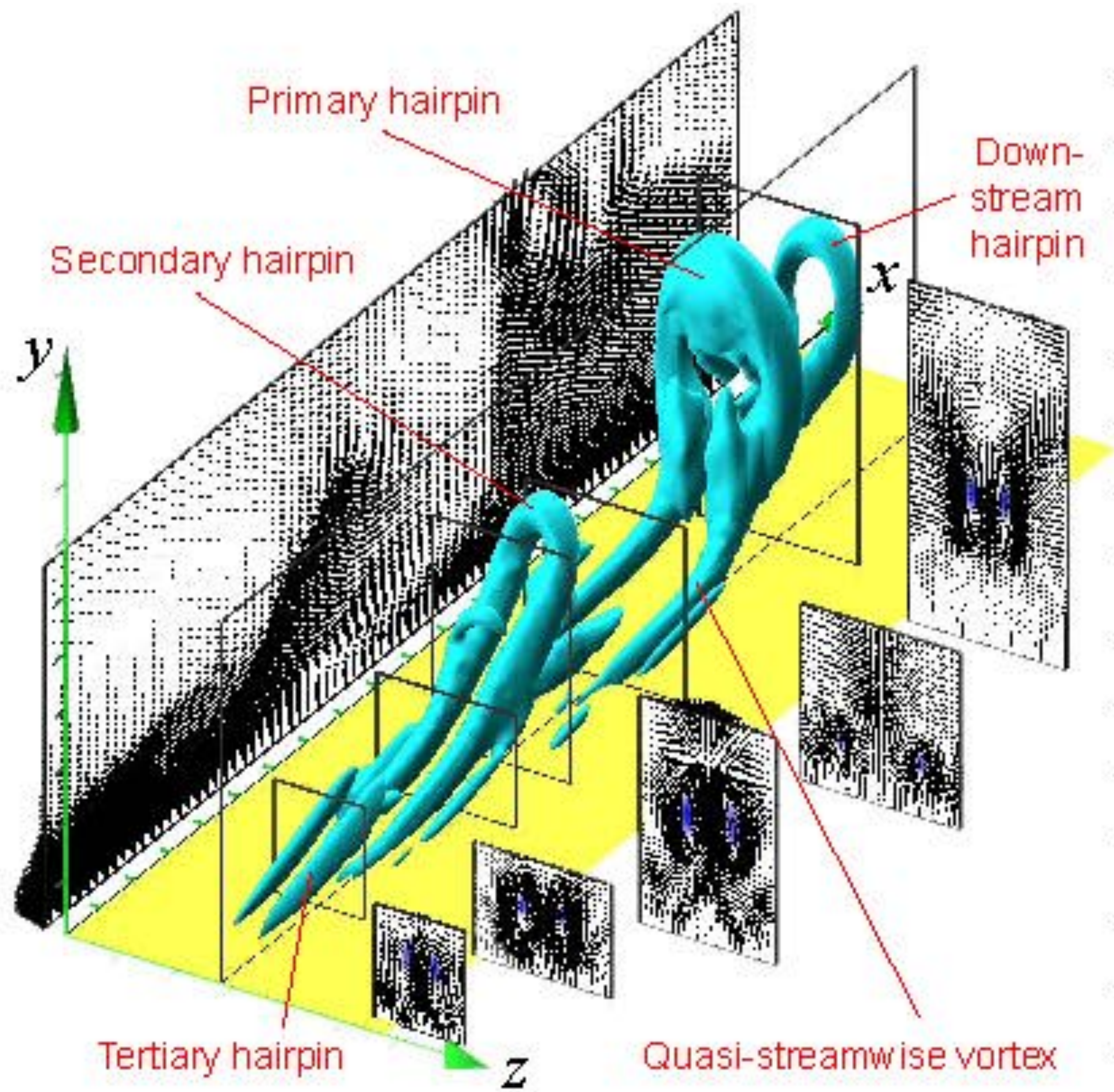
- Viscous sublayer: Alternating regions of high and low velocity (streaks).





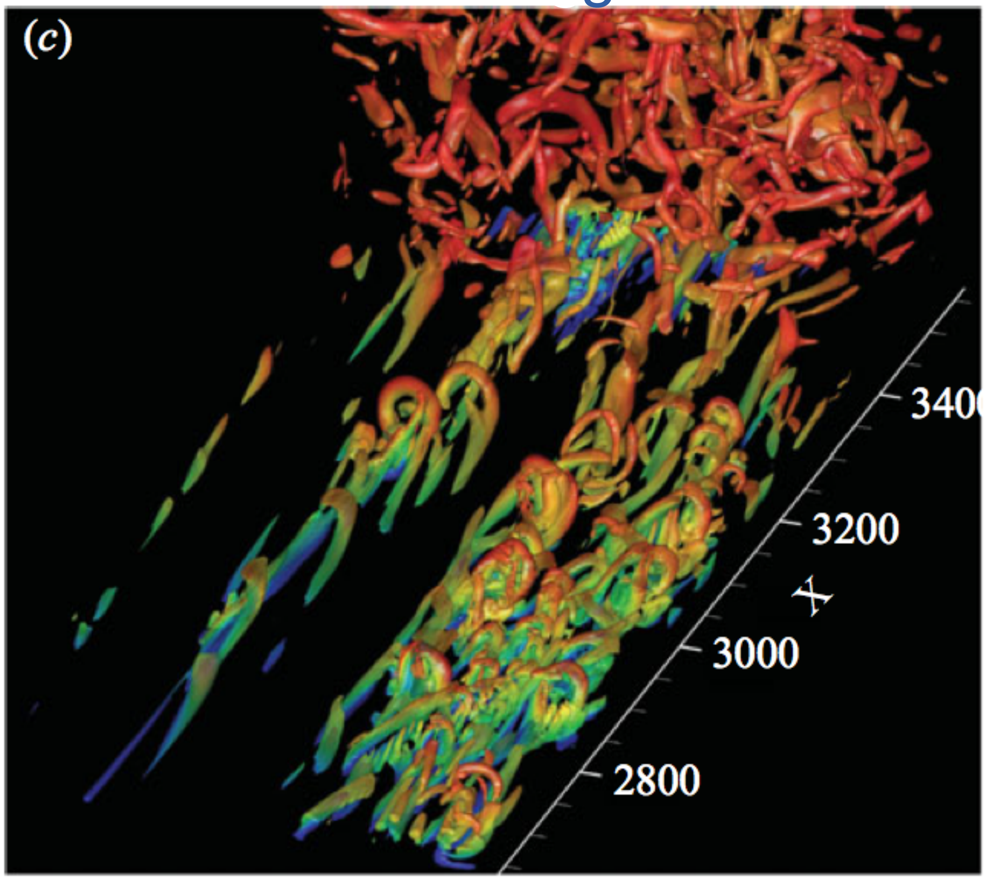
Wall-Layer Eddies

- Regions of the flow in which some variable is correlated with itself.
- Quasi-streamwise vortices (viscous sublayer).
- Hairpins (logarithmic layer).

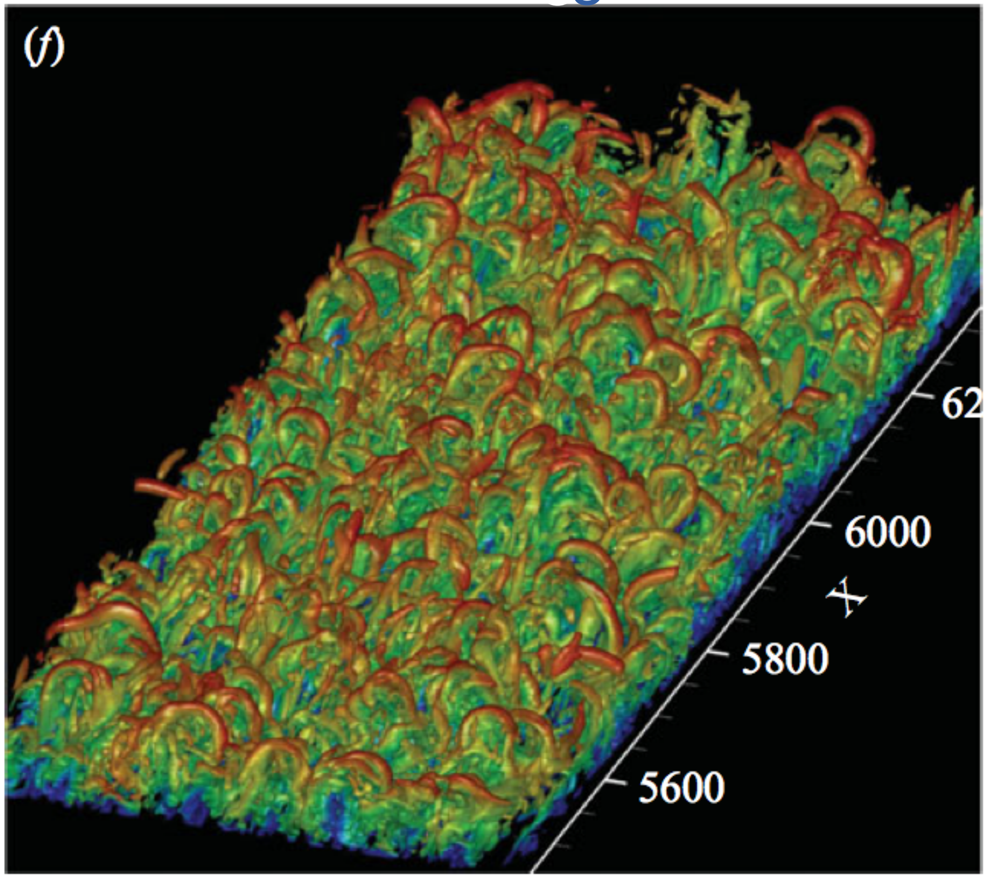


Wall-Layer Eddies

transitional region



turbulent region



DNS of transition in a flat-plate boundary layer (Wu & Moin, 2009)
Turbulent eddies are visualized by the second invariant of the velocity gradient tensor and coloured base on the local value of the streamwise velocity.

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  - *Methodologies*
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- Governing equations for LES
- Boundary conditions
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Why Simulations?

- Prediction and design tools are required
- Theory: **infeasible**
  - *Governing equations (Navier-Stokes equations) are highly non-linear.*

Conservation of mass

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0$$

Conservation of momentum

$$\frac{\partial \rho u_i}{\partial t} + + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(2S_{ij} - \frac{2}{3}\delta_{ij}S_{kk}\right)\right]$$

$$\frac{\partial e}{\partial t} + \frac{\partial}{\partial x_j}(u_j e) = \frac{\partial Q}{\partial t} - \frac{\partial}{\partial x_j}\left(\kappa\frac{\partial T}{\partial x_j}\right)$$

Conservation of energy

$\Rightarrow$ *Exact solutions cannot be found*

Why Simulations?

- Prediction and design tools are required
- Theory: **infeasible**
- Experiments
  - *Empirical design rules*
  - *Building and testing of prototypes*
  - *Iterative improvement of design*
  - *Issues: time, cost, accessibility of conditions, limited exploration*

Why Simulations?

- Prediction tools are required
- Theory: **infeasible**
- Experiments: **costly**, **incomplete**
- Numerical methods:
  - *Predict performance of proposed designs*
  - *Advantages: speed, novel designs, optimization*
  - *Issues: accuracy, reliability, level of description, cost*

Why Simulations?

- Prediction tools are required
- Theory: **infeasible**
- Experiments: **costly**, **incomplete**
- Numerical methods: **possible but difficult**

- Trend: more use of computation, less testing.

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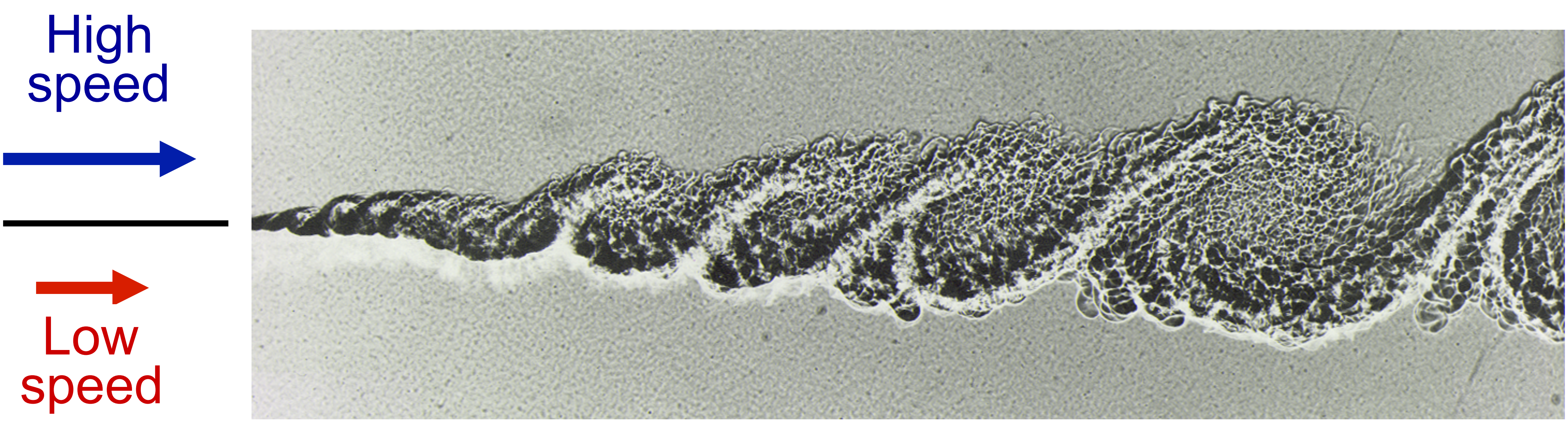
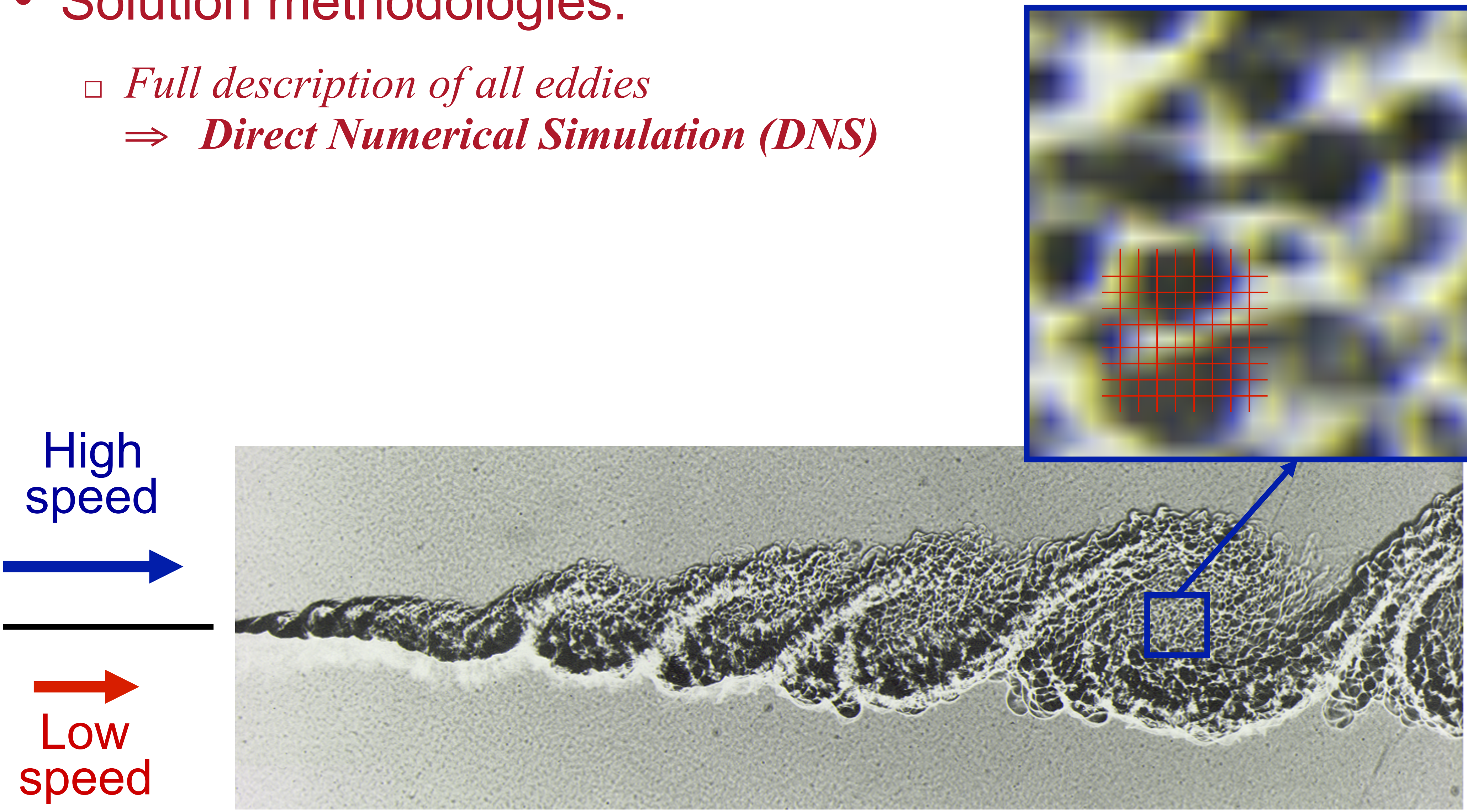
Simulation Methodologies

- Turbulent transport is due to the vortical motions (eddies).



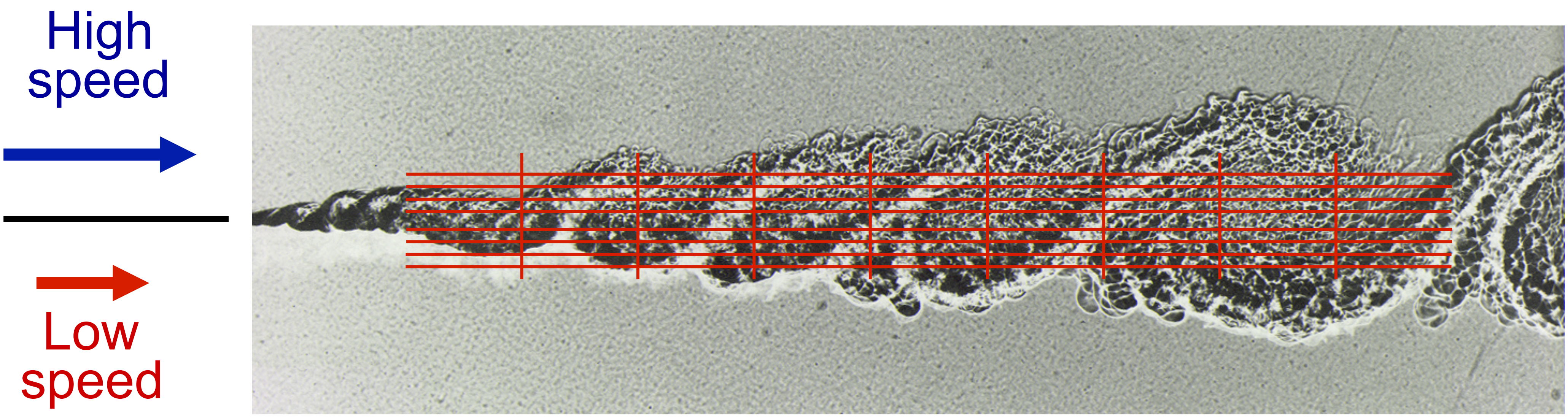
Simulation Methodologies

- Turbulent transport is due to the vortical motions (eddies).
- Solution methodologies:
  - *Full description of all eddies*
    ⇒ ***Direct Numerical Simulation (DNS)***



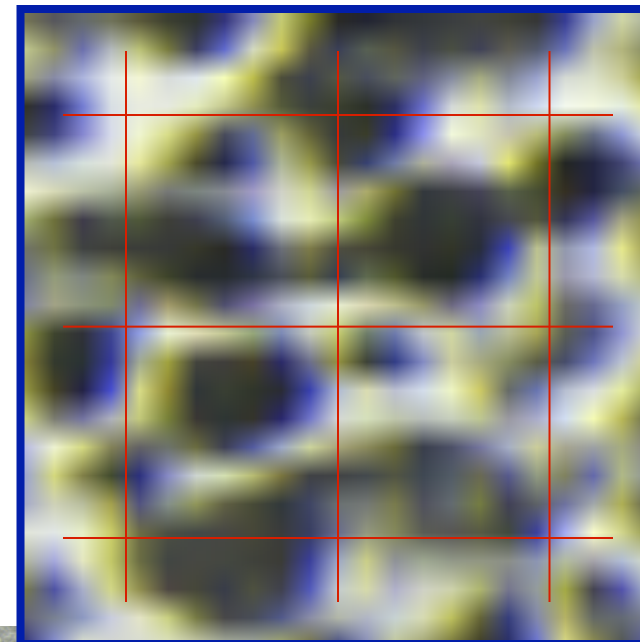
Simulation Methodologies

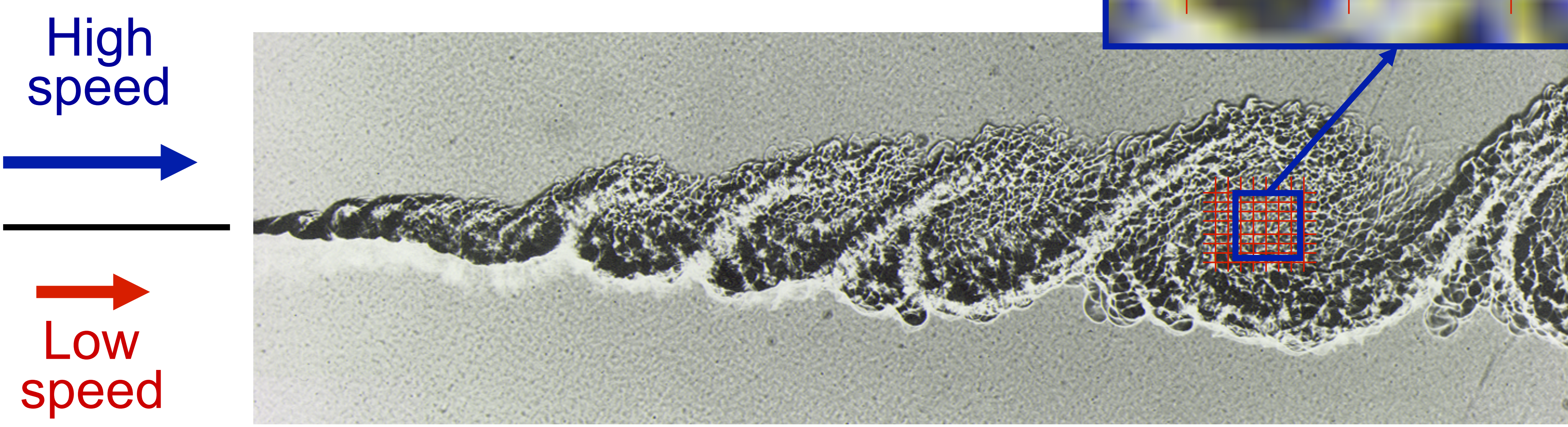
- Turbulent transport is due to the vortical motions (eddies).
- Solution methodologies:
  - *Full description of all eddies (DNS)*
  - *Statistical description of all eddies*
    ⇒ ***Solution of the Reynolds-Averaged Navier-Stokes (RANS) equations***



Simulation Methodologies

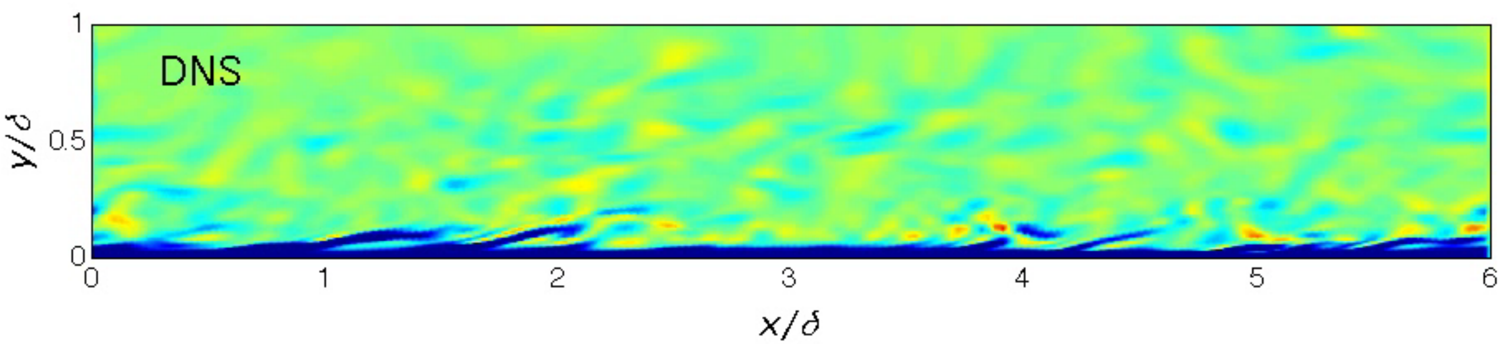
- Turbulent transport is due to the vortical motions (eddies).
- Solution methodologies:
  - *Full description of all eddies (DNS)*
  - *Statistical description of all eddies (RANS)*
  - *Partial description of the eddies*
    $\Rightarrow$ ***Large-Eddy Simulation (LES)***





Direct Numerical Simulation

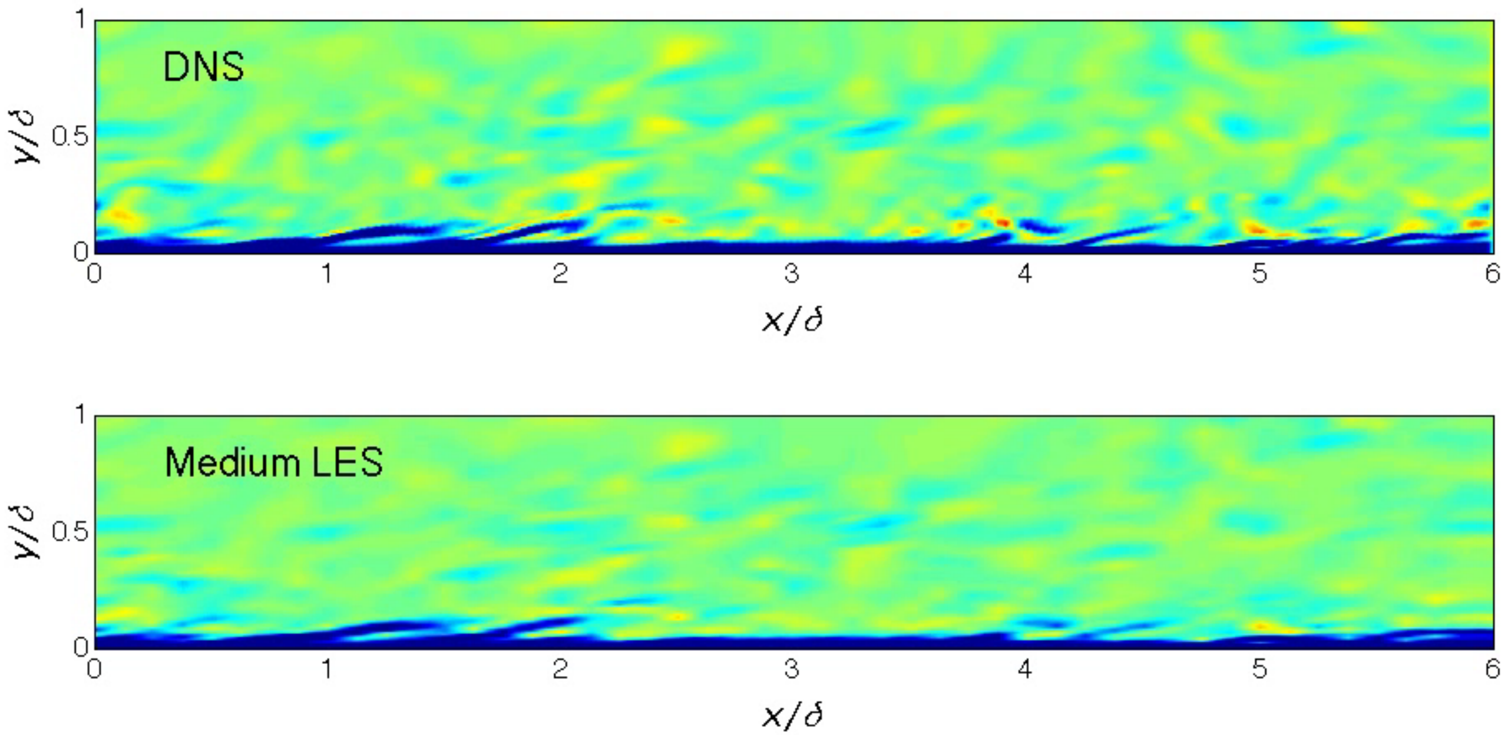
Navier-Stokes Equations → DNS Solver



Vorticity contours, Channel flow, Re=7000

Large-Eddy Simulation

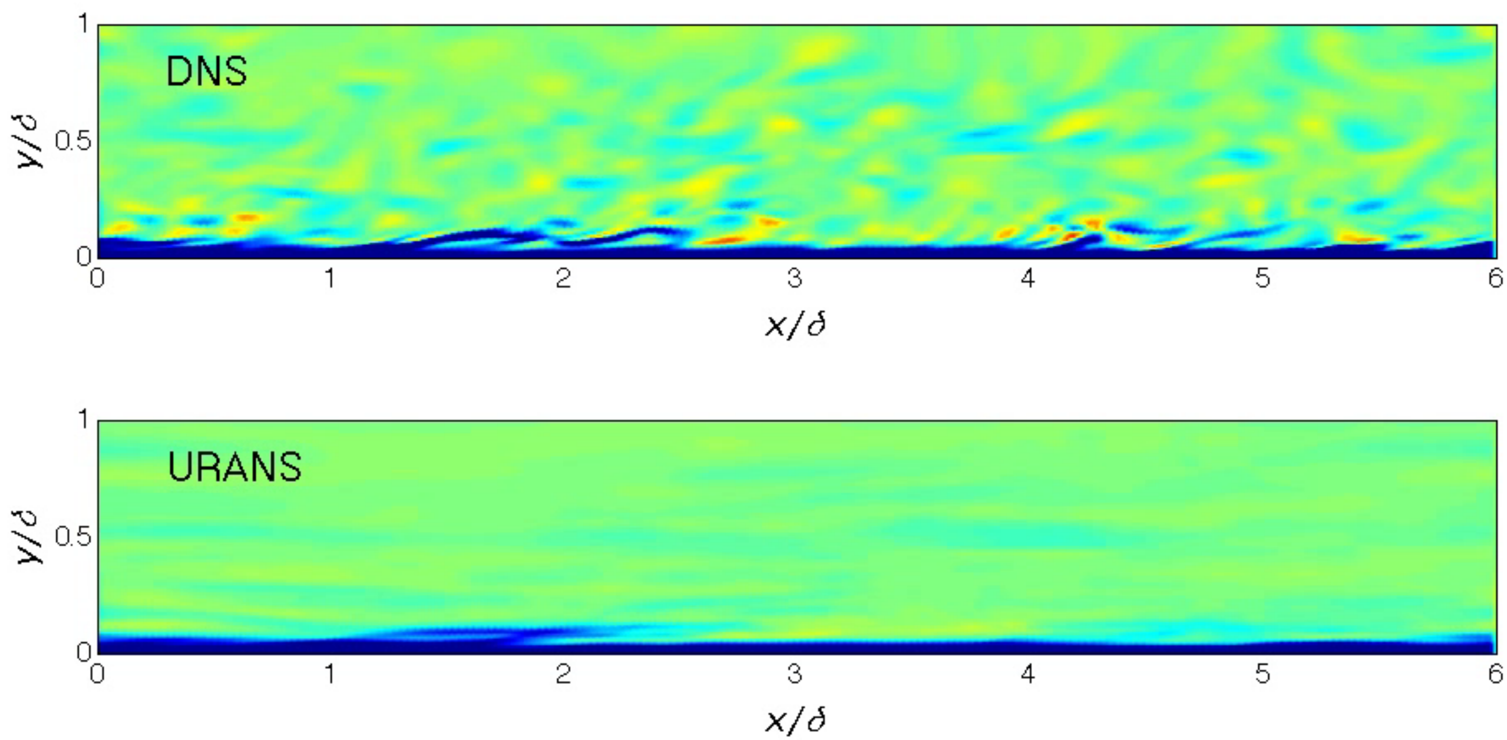
Navier-Stokes Equations → Spatial filtering over $L >> \Delta >> \eta$ → LES Solver



Vorticity contours, Channel flow, Re=7000

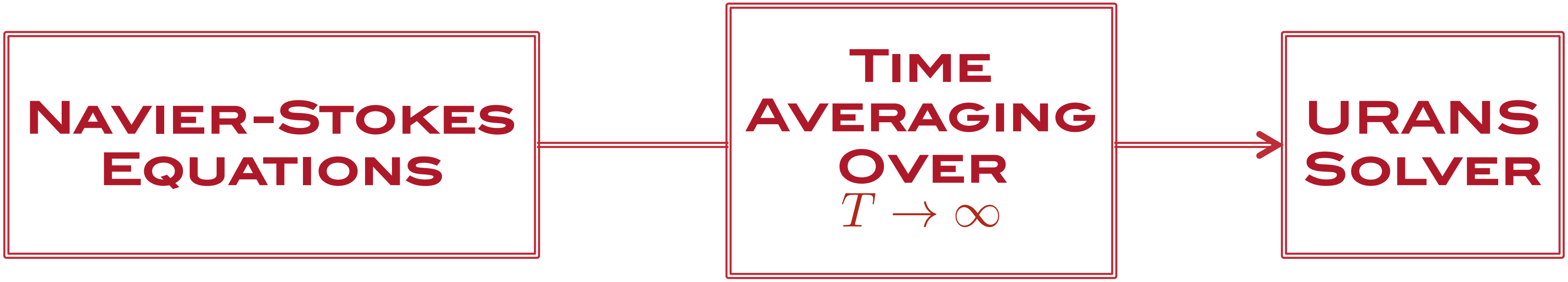
Unsteady Reynolds-Averaged NS Simulation

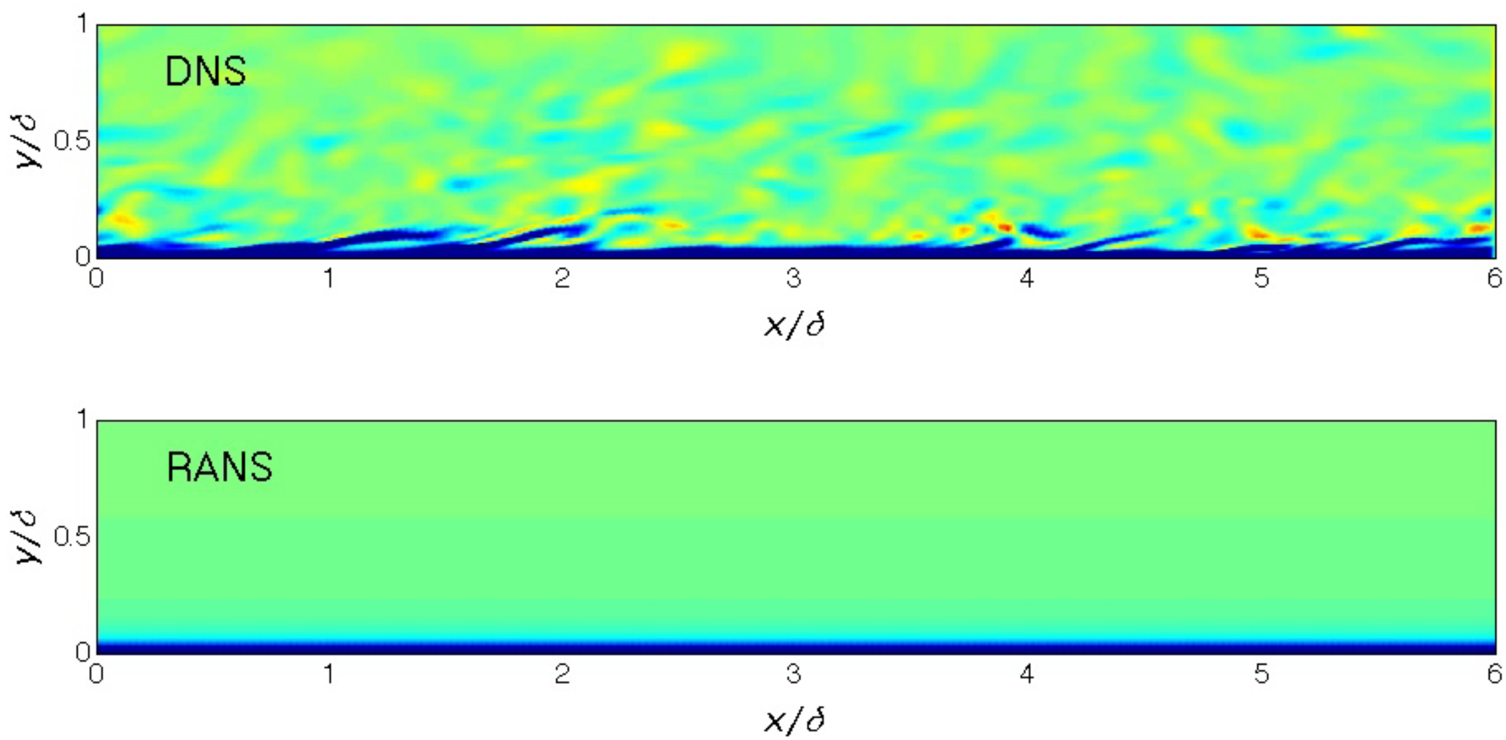
Navier-Stokes Equations → Time Averaging Over $T >> LETOT$ → URANS Solver



Vorticity contours, Channel flow, Re=7000

Unsteady Reynolds-Averaged NS Simulation





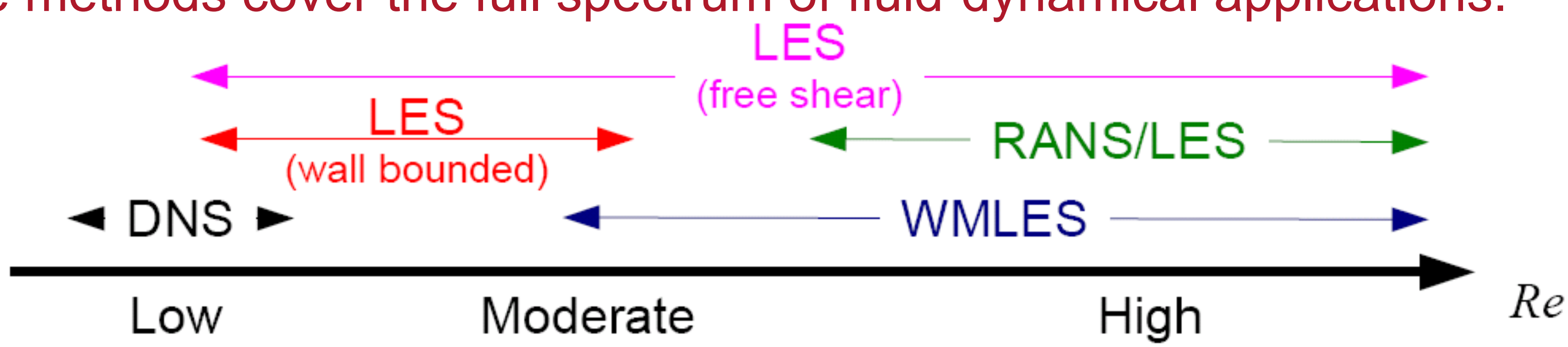
Vorticity contours, Channel flow, Re=7000

Why LES and DNS?

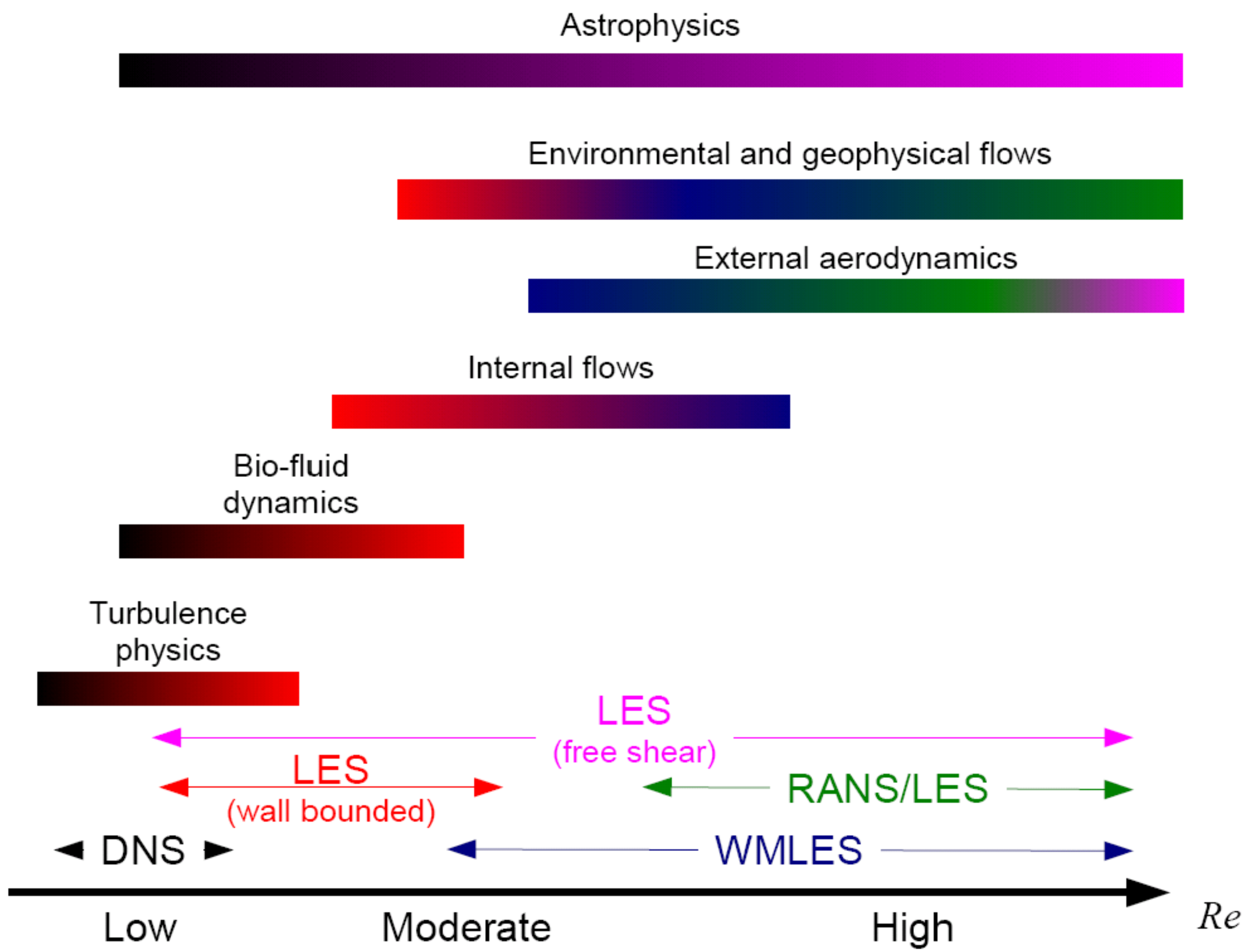
- Multi-point, non invasive information.
- Frequency and wave-number information.
- Little modelling ⇒ increased accuracy
- More faithful reproduction of the flow physics.

Benefits of Eddy-Resolving Methods

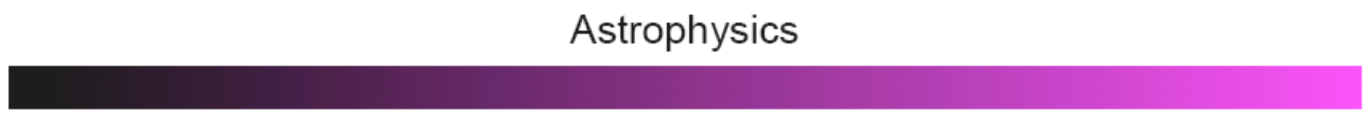
- Eddy-resolving methods for the numerical simulation of turbulent flows have resulted in
  - *Improved understanding of the flow physics*
  - *Novel flow-control ideas*
- Direct Numerical Simulations (DNS):
  - *No empiricism, Low Re, physics, simple geometry.*
- Large-Eddy Simulations (LES):
  - *Little empiricism, medium Re, physics.*
- Hybrid RANS/LES:
  - *Stronger empiricism, high Re, physics and design, complex geometry.*
- These methods cover the full spectrum of fluid-dynamical applications.



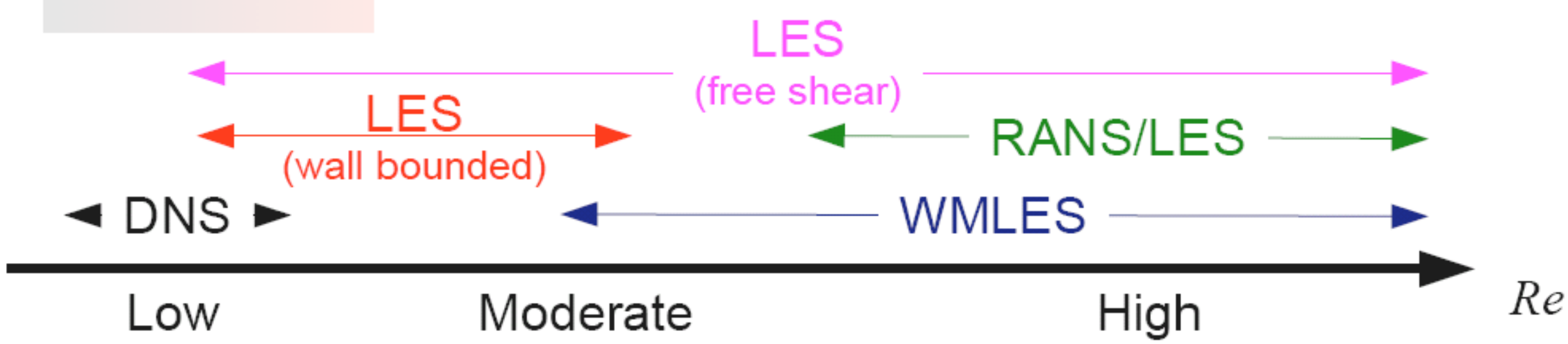
Benefits of eddy-resolving methods



Benefits of Eddy-Resolving Methods

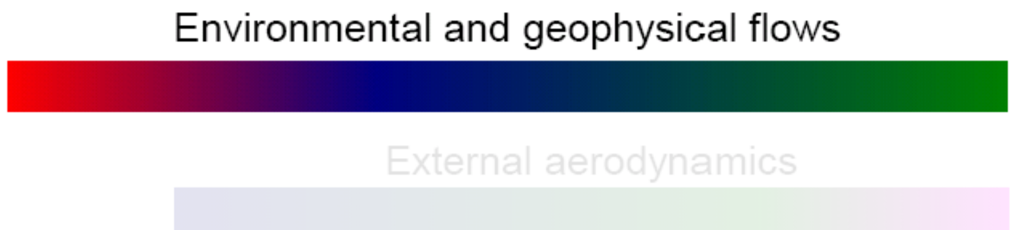


- Range of scales: $10^{10} \Rightarrow$ DNS is infeasible.
- Wall effects are negligible
  $\Rightarrow$ LES can be cost-effective at high Re
- Interaction of large-scale rotation with turbulent eddies.
- Lorentz forces are an additional mechanism that affects the turbulence dynamics.
  - *Stiffness*

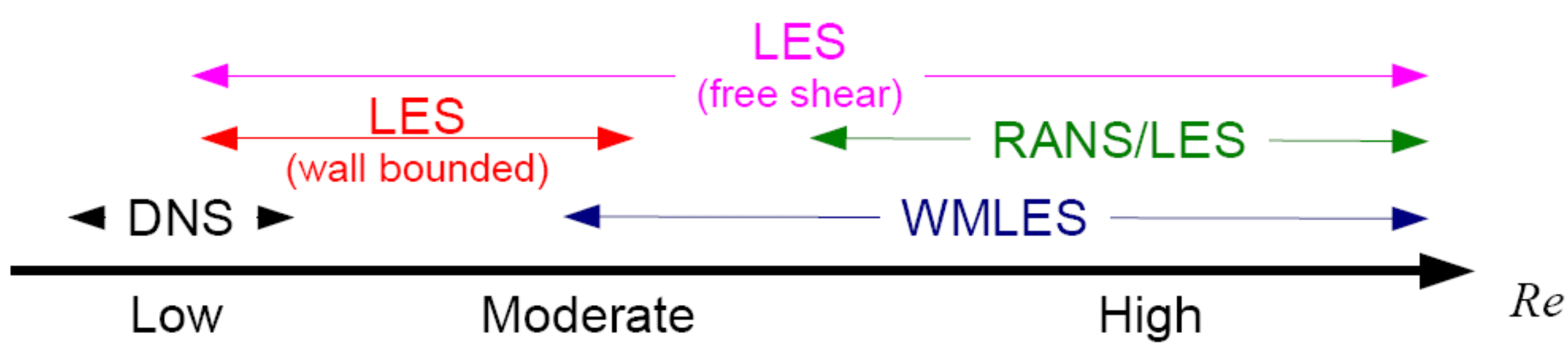


Benefits of Eddy-Resolving Methods

- Eddy-resolving methods can account directly for
  - *Large-scale mixing*
  - *Stratification*
  - *Rotation effects*

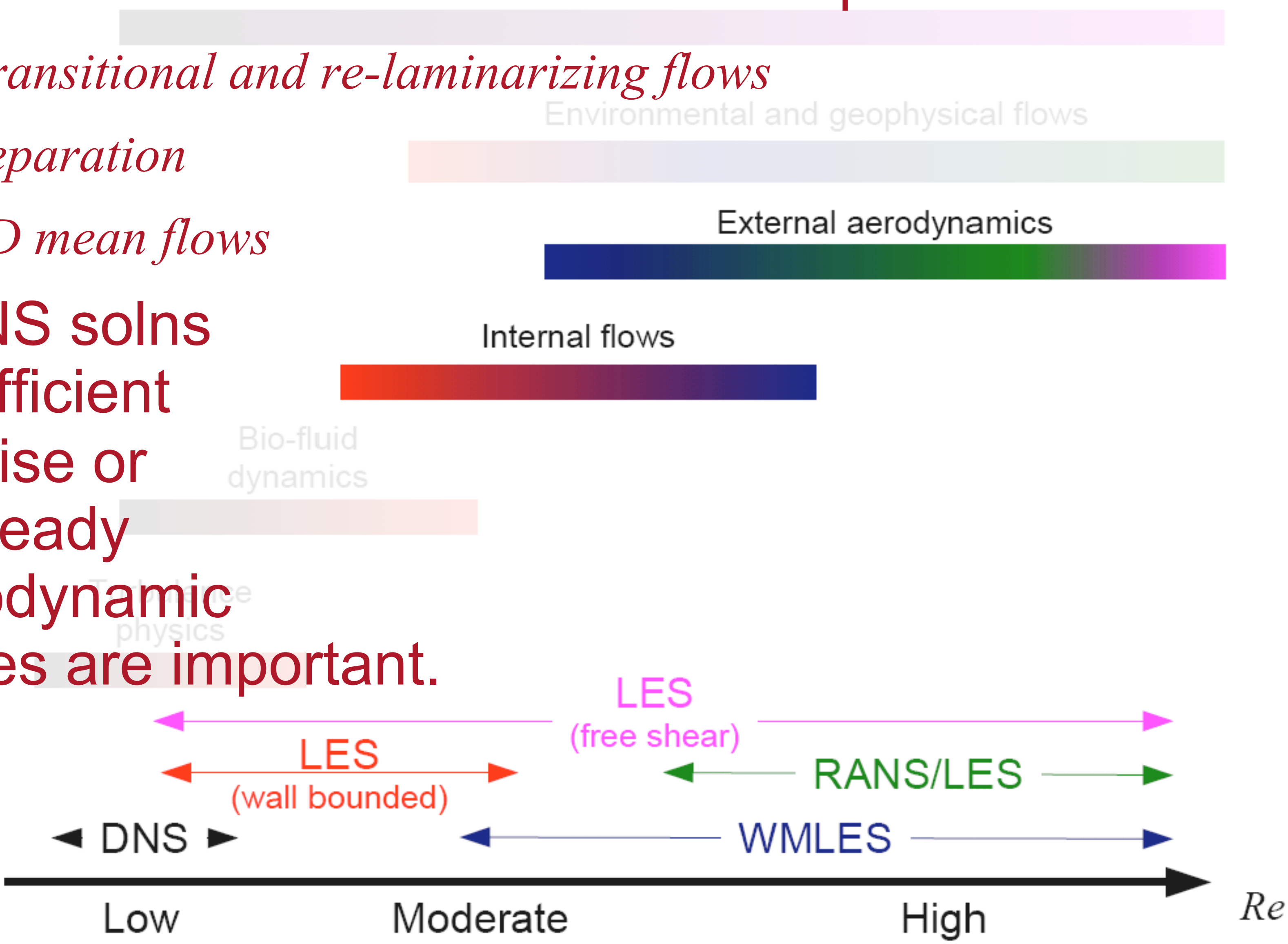


- Possible applications:
  - *Sediment transport*
  - *Coastal oceanography*
  - *Pollutant dispersion*



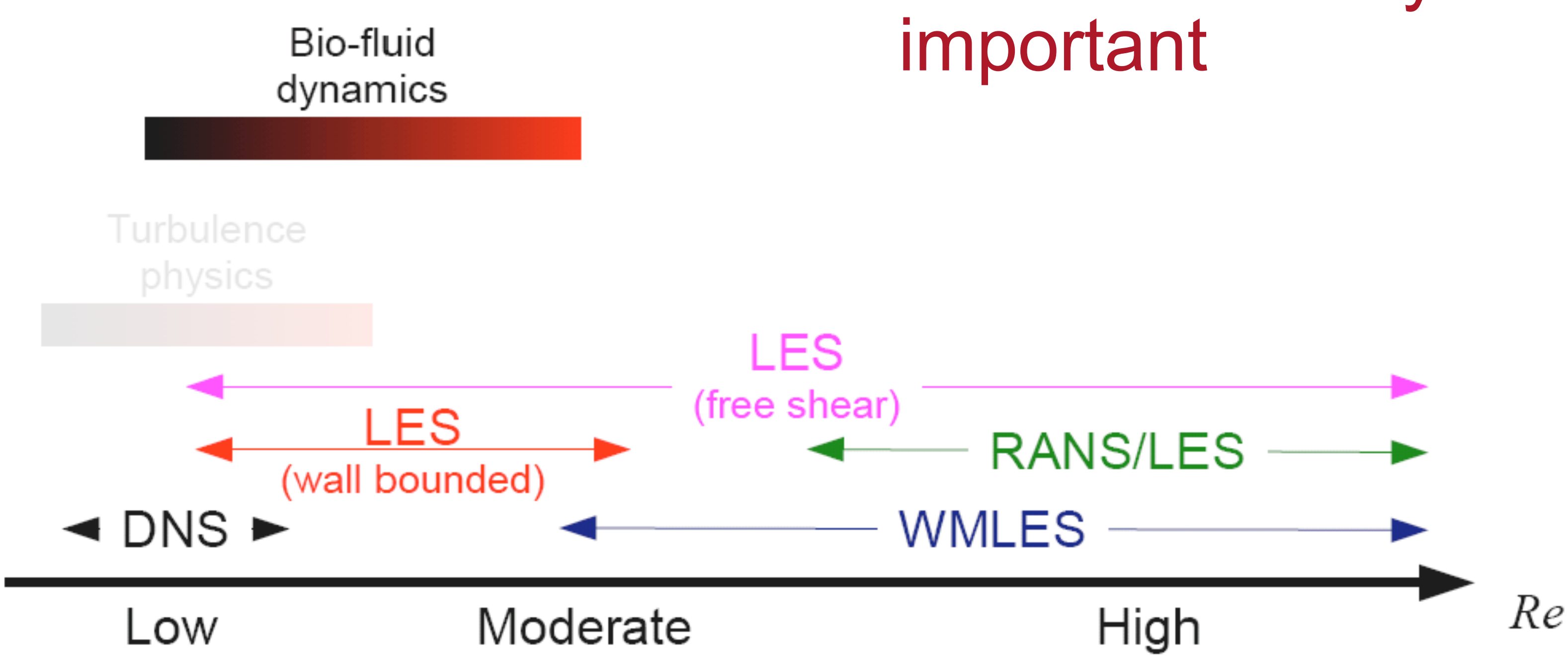
Benefits of eddy-resolving methods

- More accurate calculation of non-equilibrium turbulence:
  - *Transitional and re-laminarizing flows*
  - *Separation*
  - *3D mean flows*
- RANS solns insufficient if noise or unsteady aerodynamic forces are important.



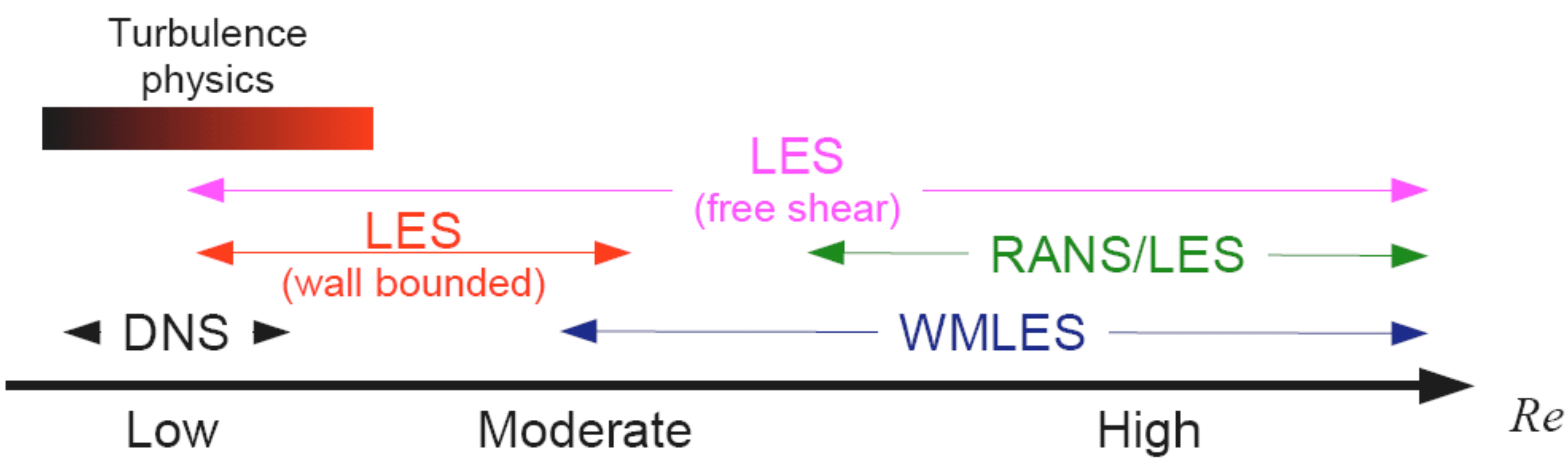
Benefits of eddy-resolving methods

- Moderate Re, complex geometries
- Highly 3D, unsteady, transitional flows.
  - *RANS methods inaccurate*
- Oscillating mean flows
  - *Alternating favorable and adverse VP*
    ⇒ Relaminarization and transition
- Fluid-structure interactions may be important



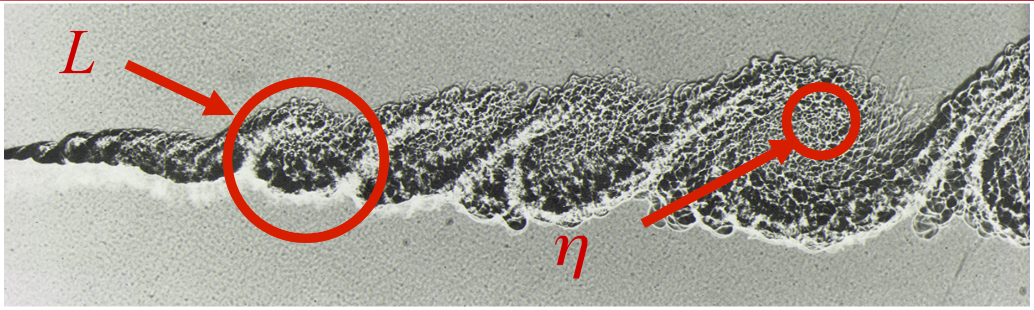
Benefits of eddy-resolving methods

- Ideal method to test theories
  - *Controlled boundary and initial conditions.*
  - *Possibility to conduct innovative thought experiments*
- Data can be used to test lower-level models
  - *Understanding of the physical phenomena involved*
  - *Determination of constants*
  - *Term-by-term model evaluation*



- **Motivation:**
  - ***What is turbulence?***
  - ***Review of turbulence physics***
  - ***Why simulations?***
  - ***Methodologies***
  - ***Resolution requirements***
- Governing equations for LES
- Boundary conditions
- Subfilter-scale modelling
- Validation of an LES
- Applications
- Challenges
- Conclusions

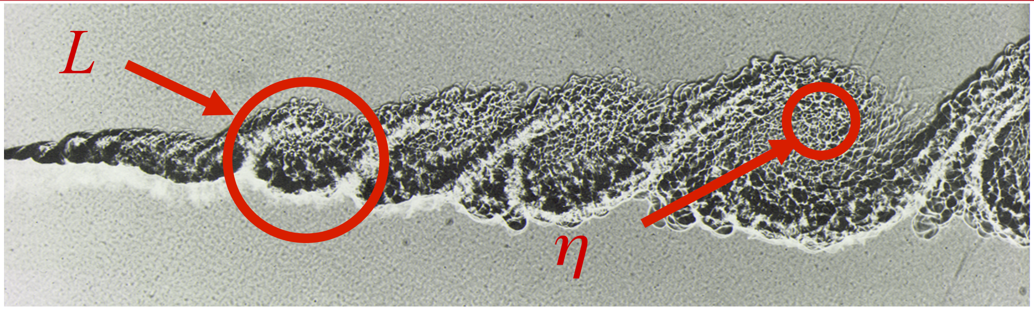
Resolution Requirements (DNS)



- The computational domain size is of the order of the integral scale L.
- The grid size must be of the order of the Kolmogorov length η

$$N_x N_y N_z \sim (L/\eta)^3 \sim Re^{9/4}$$

Resolution Requirements (DNS)



- The computational domain size is of the order of the integral scale L.
- The grid size must be of the order of the Kolmogorov length η

$$N_x N_y N_z \sim (L/\eta)^3 \sim Re^{9/4}$$

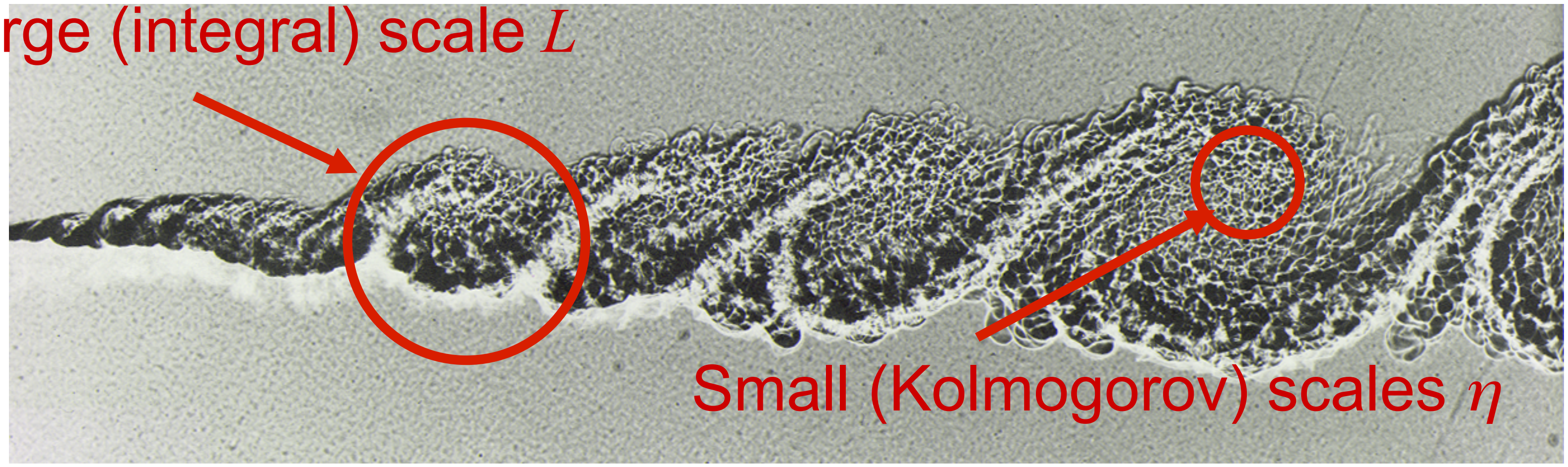
- The equations must be integrated for a time of the order of the integral time scale T.
- The time-step $\Delta T \propto$ the grid size (CFL condition) or $\propto$ the Kolmogorov time scale τ_η

$$N_t \sim T/\Delta t \sim L/\eta \sim Re^{3/4}$$

$$N_t \sim T/\tau \sim Re^{1/2}$$

$$\Rightarrow N_x N_y N_z \times N_t \sim Re^{11/4} \to Re^3$$

Resolution Requirements (DNS)



- Cost $N_x N_y N_z \times N_t \sim Re^{11/4} \rightarrow Re^3$
- $Re = 10^4 \Rightarrow$ o(10^3) CPU hours, Gflop machine
- $Re = 10^9 \Rightarrow$ o(10^{11}) CPU hours, Tflop machine

Resolution Requirements (LES)

- Velocity decomposed into large-scale (resolved) and subfilter-scale (unresolved) parts.
  - *The large scales (~ Integral scale, L), which depend on the boundary conditions (i.e., are flow dependent) are computed.*
  - *The small scales , which are more universal (less dependent on boundary conditions) are modelled.*
  - *Large scales contribute most of the Reynolds stresses.*

Resolution Requirements (LES)

- Only resolve the integral length scale (energy-containing eddies)
- The integral scale varies with Re.
- Cost scales with the Reynolds number

$\Rightarrow$ **High Re calculations are possible.**

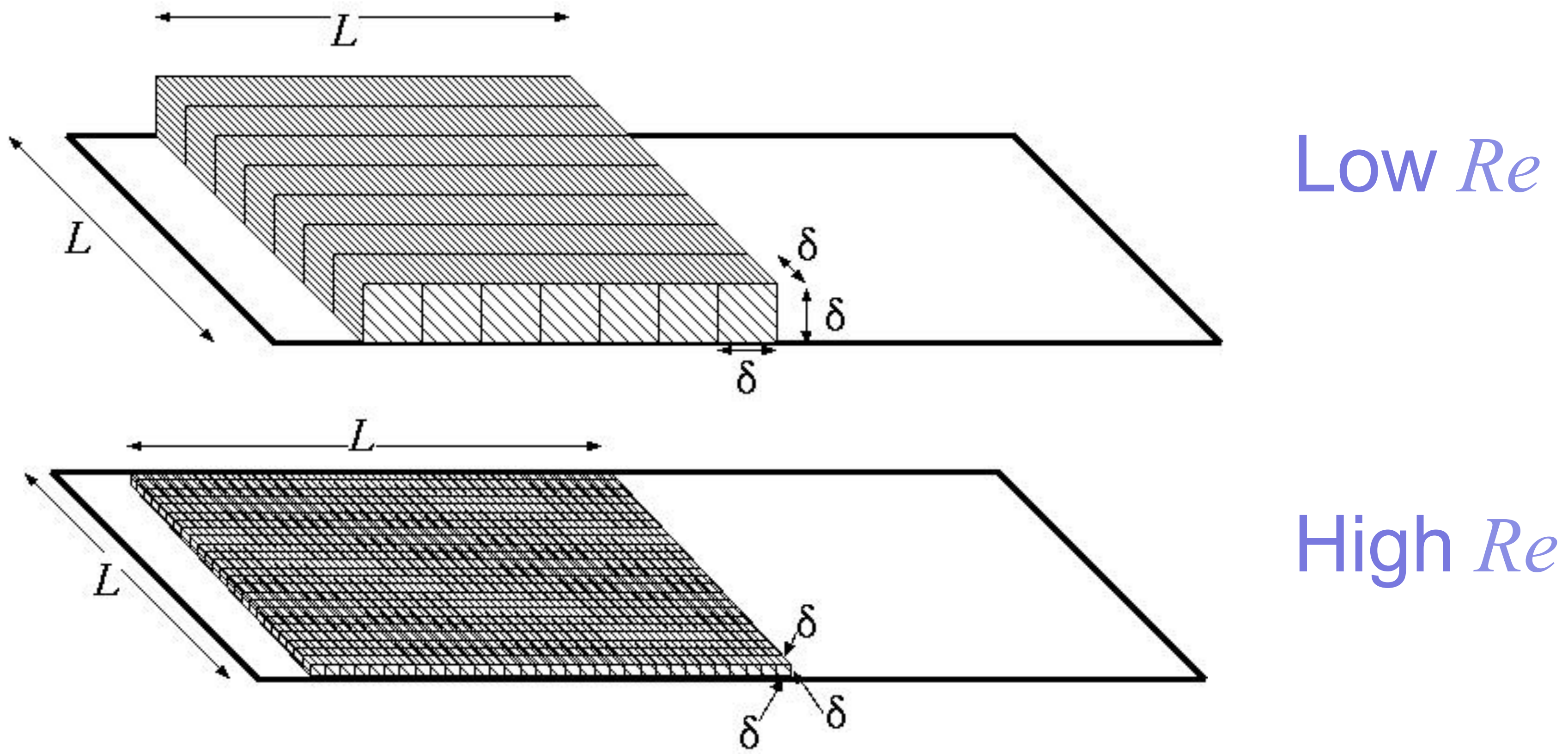
Resolution Requirements (Wall-bounded flows)

- Outer layer:
  - *Need 20-30 points in each direction to resolve an integral scale δ.*
  - *Cover the body of dimensions L^2 with a layer of cubes of side δ.*

$$N_x N_y N_z \sim N_{cubes} \sim (L/\delta)^2$$

Resolution Requirements (Wall-bounded flows)

- Outer layer:
  - *L/δ varies with Re*



Resolution Requirements (Wall-bounded flows)

- Outer layer:
  - *L/δ varies with Re slowly (generally $\sim Re^{0.2}$)*

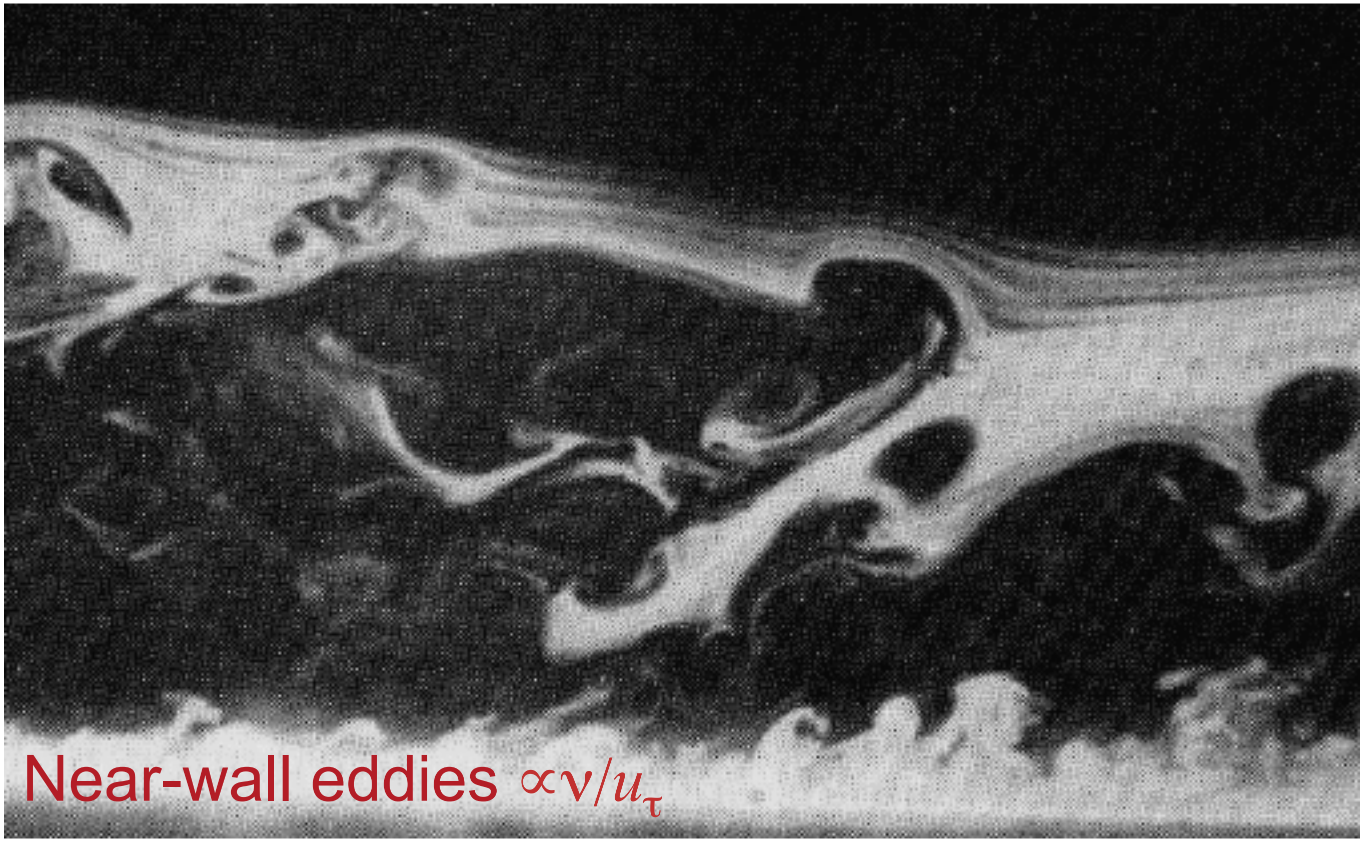
$$N_x N_y N_z \sim N_{cubes} \sim (L/\delta)^2 \sim Re^{0.4}$$

  - *Total cost of the calculation:*

$$N_x N_y N_z \times N_t \sim Re^{0.6}$$

Resolution Requirements (Wall-bounded flows)

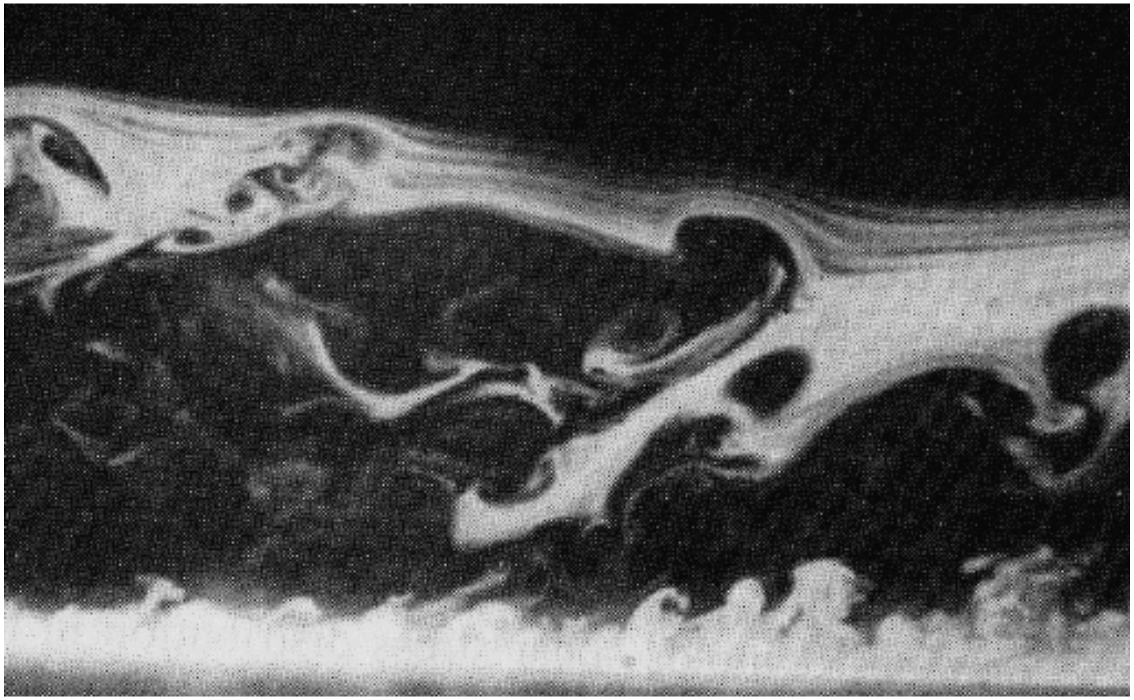
- Outer layer: only scales of order δ must be resolved.
  $\Rightarrow$ Cost ~ $Re^{0.6}$
- Inner layer: near-wall eddies must be resolved.



Resolution Requirements (Wall-bounded flows)

- Outer layer: only scales of order δ must be resolved.
  $\Rightarrow$ Cost ~ $Re^{0.6}$
- Inner layer: near-wall eddies must be resolved.
  - *Grid must scale in wall units*
  - $N_x N_y N_z \propto Re^{1.8}$

$\Rightarrow$ Cost $\propto Re^{2.4}$



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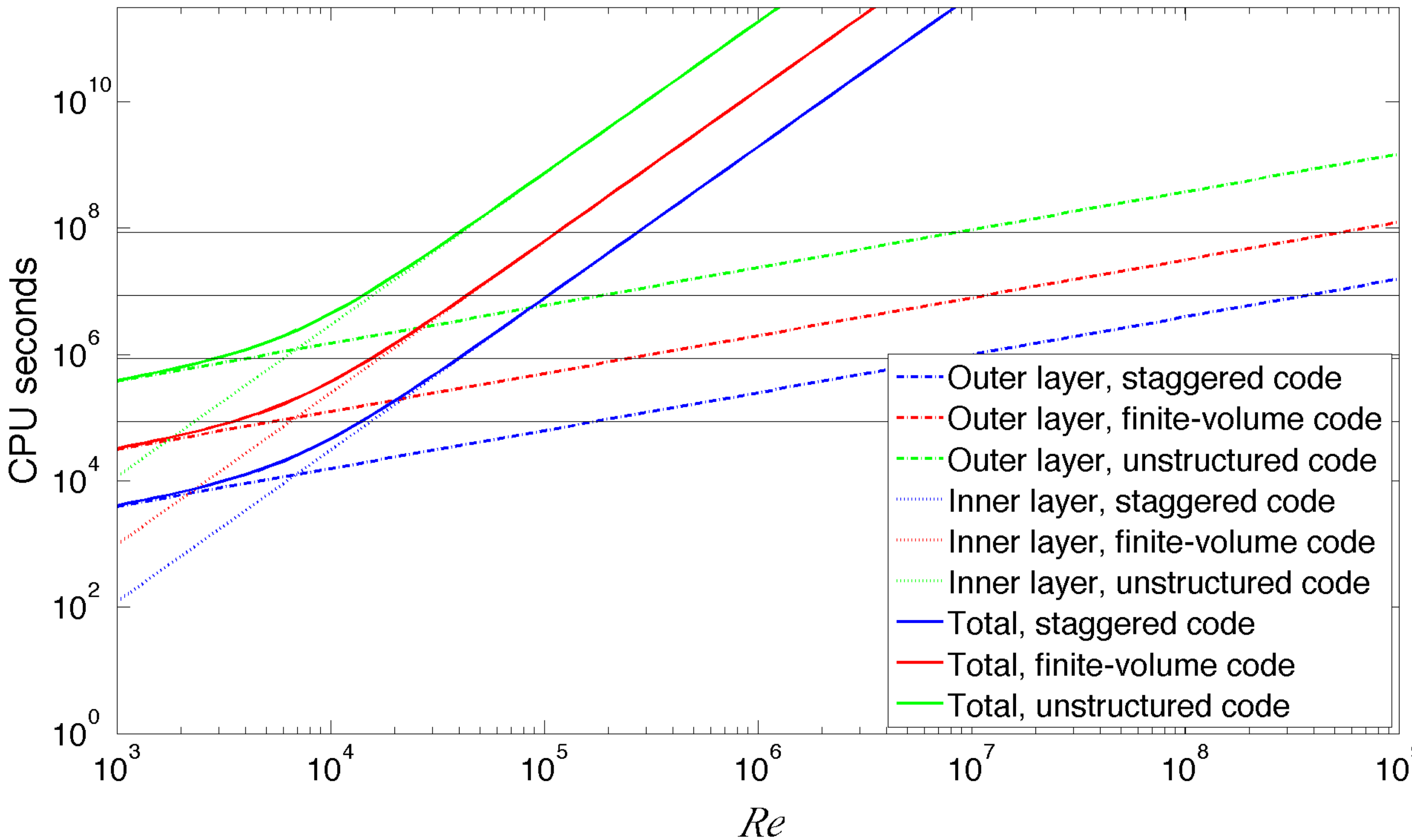

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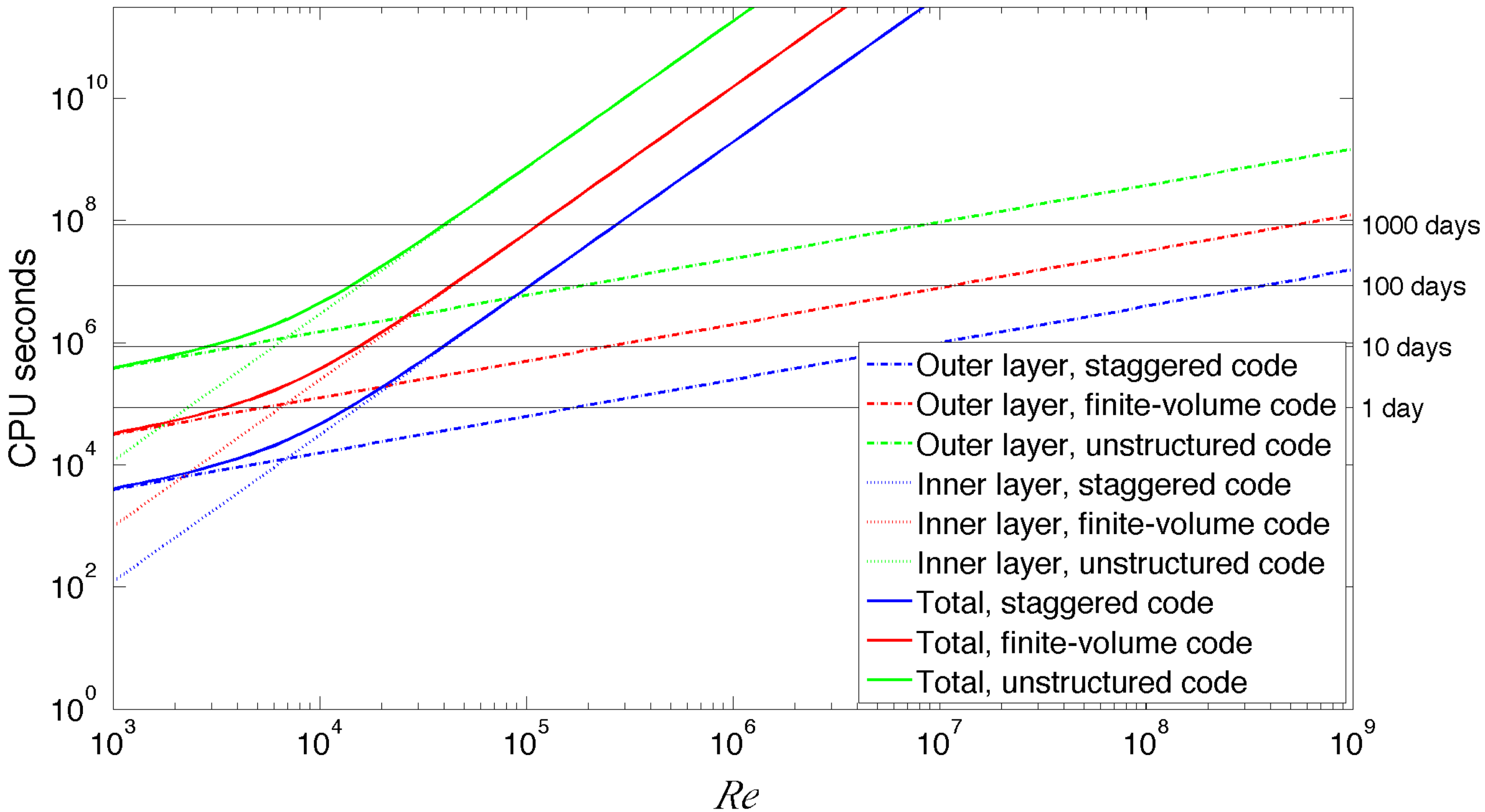
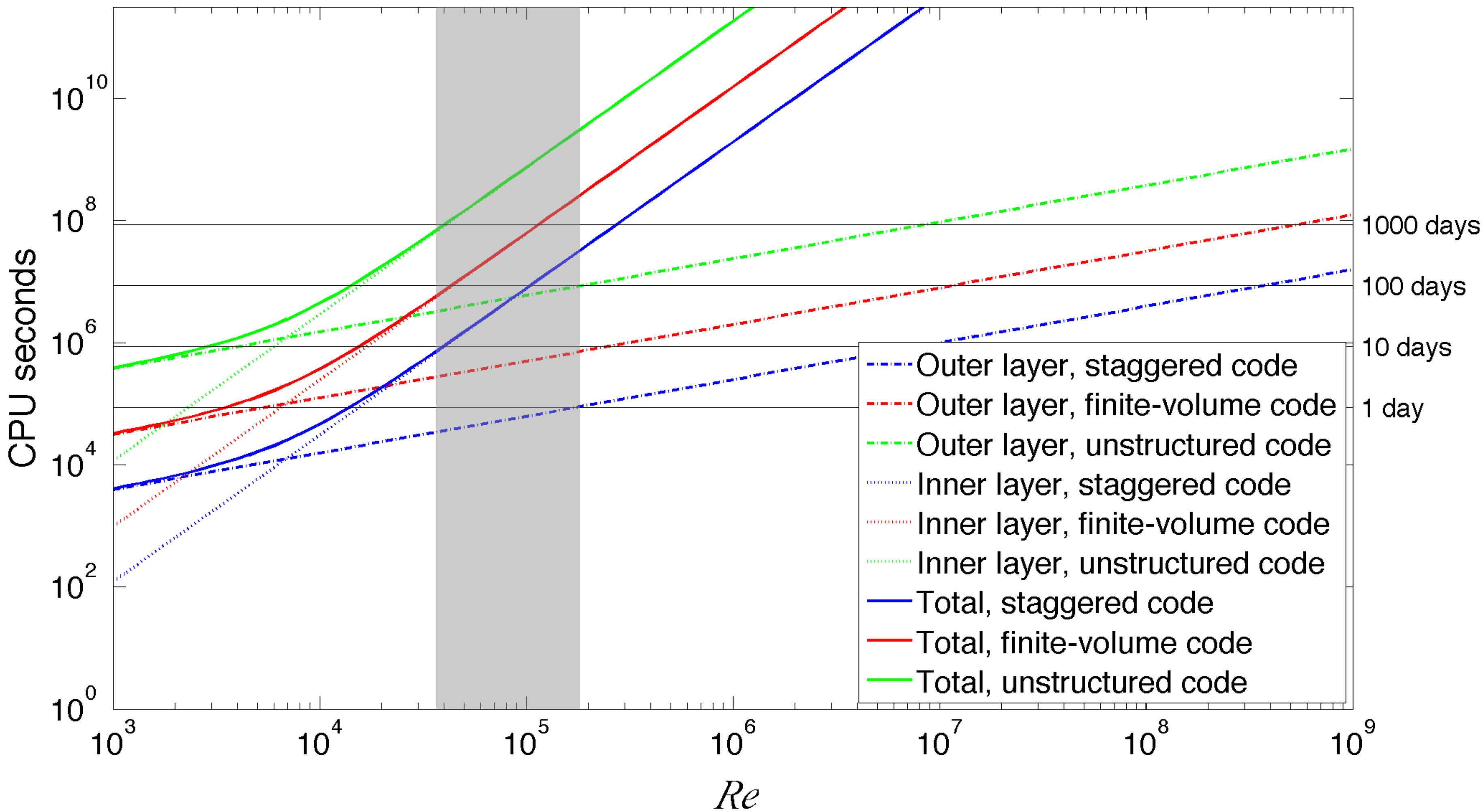


Resolution Requirements (Wall-bounded flows)



$\Rightarrow$ At high Re, throughput times are too long for design applications

Why Not LES and DNS?

- Small scales need to be resolved
  $\Rightarrow$ fine grid.
- Averaging of the results, not of the equations
  $\Rightarrow$ long integration times.
- Vortex dynamics are important, vortex stretching must be accounted for
  $\Rightarrow$ always 3D.

$\Rightarrow$ **Require large computational resources**

Outline

- Motivation:
- Simulation methodologies
- **Governing equations**
- Numerical methods
- Subfilter-scale modelling
- Validation of an LES
- Applications
- Challenges
- Conclusions

Outline

- Motivation:
- Simulation methodologies
- **Governing equations**
  - ***Filtering***
  - *Filtered Navier-Stokes equations*
- Numerical methods
- Subfilter-scale modelling
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Conservation equations

- Conservation of mass, momentum and energy

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) &= 0 \\
\frac{\partial \rho u_i}{\partial t} + + \frac{\partial}{\partial x_j}(\rho u_i u_j) &= -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(2S_{ij} - \frac{2}{3}\delta_{ij}S_{kk}\right)\right] \\
\frac{\partial e}{\partial t} + \frac{\partial}{\partial x_j}(u_j e) &= \frac{\partial Q}{\partial t} - \frac{\partial}{\partial x_j}\left(\kappa \frac{\partial T}{\partial x_j}\right),
\end{aligned}$$

- Where:

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$ Strain-rate tensor

$$e = c_v T + \rho u_i u_i$$ Total energy

$$\kappa = \alpha / \rho c_p$$ Diffusivity

Conservation equations (Incompressible flow)

- If ρ = constant

$$\frac{\partial u_j}{\partial x_j} = 0$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j} u_i u_j = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i$$

$$\frac{\partial T}{\partial t} + \frac{\partial}{\partial x_j} u_j T = \kappa \nabla^2 T$$

Governing equations

- Velocity decomposed into large-scale (resolved) and subfilter-scale (unresolved) parts.
  - *The large scales (~* ***Integral scale, L****), which depend on the boundary conditions (i.e., are flow dependent) are computed.*
  - *The small scales , which are more universal (less dependent on boundary conditions) are modelled.*
  - *Large scales contribute most of the Reynolds stresses.*

Governing equations

- The large and small scales are separated by the filtering operation

$$\overline{f}(\mathbf{x}) = \int_D f(\mathbf{x}')G(\mathbf{x},\mathbf{x}',\overline{\Delta})d\mathbf{x}'$$

- Filtering is a local spatial averaging over the filter width Δ that smooths out fluctuations whose scale is >Δ.
- Filtering in general is **not** a Reynolds operator:

$$\overline{f'} \neq 0; \quad \overline{\overline{f}} \neq \overline{f}$$

Governing equations

- The filtering operation can be applied to the NS equations

$$\int_D [\text{Navier-Stokes Eqns}] G(\mathbf{x}, \mathbf{x}', \overline{\Delta}) d\mathbf{x}'$$

to yield the filtered NS equations

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0$$

$$\frac{\partial \overline{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{u}_i \overline{u}_j\right) = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} - \boxed{\frac{\partial \tau_{ij}}{\partial x_j}} + \nu \frac{\partial^2 \overline{u}_i}{\partial x_j \partial x_j}$$

$$\frac{\partial \overline{T}}{\partial t} + \frac{\partial}{\partial x_j}\overline{u}_j \overline{T} = -\boxed{\frac{\partial Q_j}{\partial x_j}} + \kappa \nabla^2 \overline{T}$$

- Unresolved stresses and heat flux appear:

$$\boxed{\tau_{ij} = \overline{u_i u_j} - \overline{u}_i \overline{u}_j} \qquad \boxed{Q_j = \overline{u_j T} - \overline{u}_j \overline{T}}$$

GOVERNING EQUATIONS

- The unresolved stresses are known as **Subgrid-Scale (SGS) stressess** or (better) as **SubFilter-Scale (SFS) stresses.**
- They must be expressed in terms of filtered variables (SFS model)

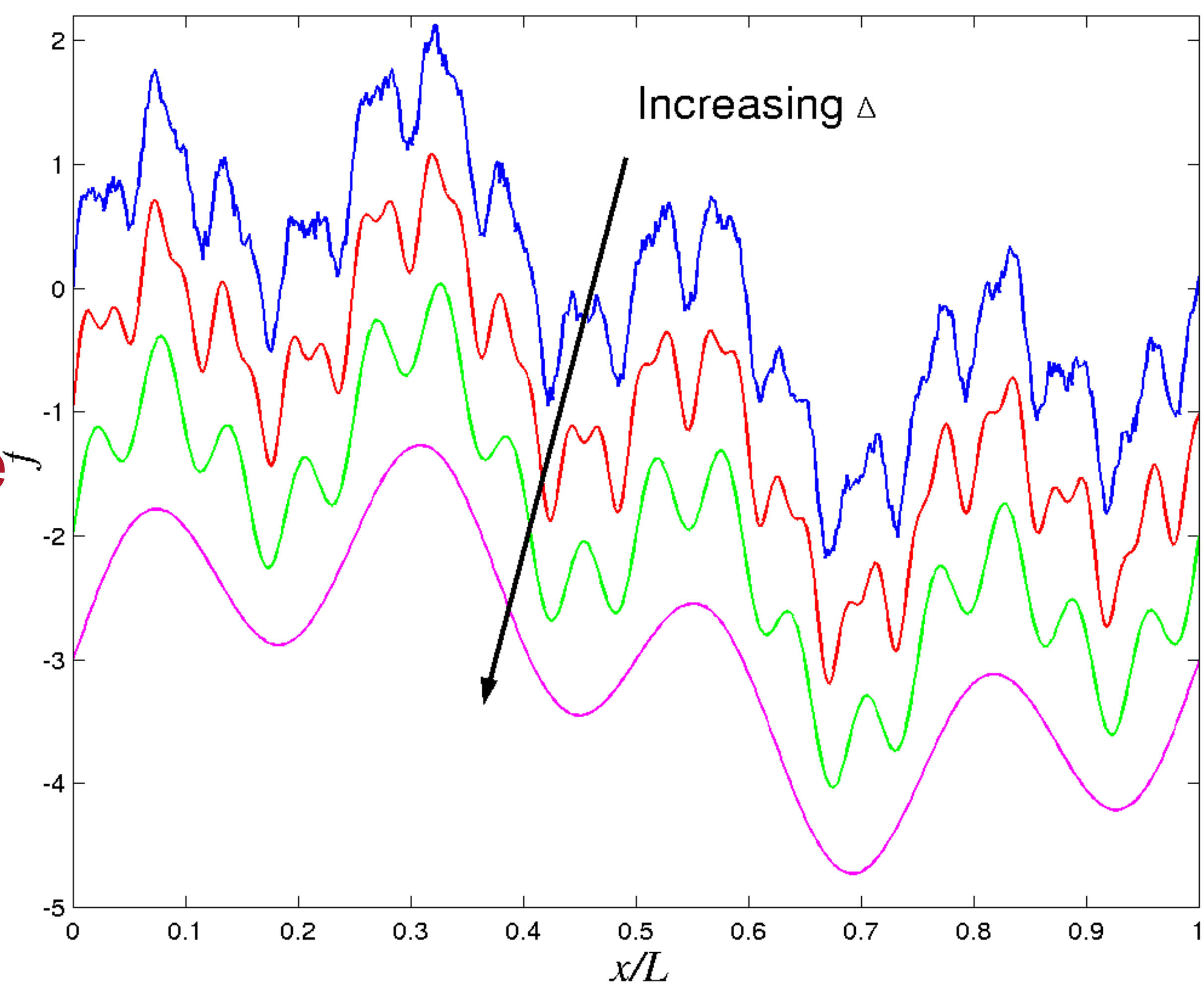
$$\tau_{ij} = f(\overline{u}_i, \overline{S}_{ij}, \ldots)$$

  - *The small scales, which are more universal (less dependent on boundary conditions) are modelled.*

- The large scales, which depend on the boundary conditions(i.e., are flow dependent) are computed.

The filtering operation

- Filtering is a local spatial averaging over the filter width Δ.
- Increasing Δ removes more and more scales from the velocity field $\Rightarrow$ the contribution of τ_{ij} increases.



Filtering considerations

- Fundamental assumption in LES:

  The energy-carrying eddies are resolved, only the small scales are modelled

- Implication:

  The filter-width must be smaller than the local integral scale, L

- Practice:

  The filter width Δ is proportional to the grid size h

Filtering considerations

- If Δ is proportional to h:
  - *The ratio between filter-width and integral scale varies*
    - ⇨ Suboptimal grids (that are not refined when the integral scale decreases) may have unexpectedly large errors
  - *Rapid variations of the grid size are reflected in the eddy viscosity*
    - ⇨ Unphysical discontinuities in the SFS contribution to the transport can occur.
    - ⇨ Numerical and commutation errors may be significant.
  - *Grid convergence studies are difficult.*

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