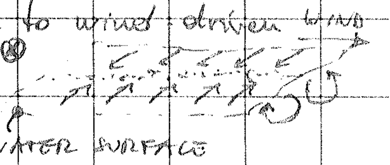
PARTICLE TRACKING IN A CELLULAR FLOW FIELD

Objective : simulate numerically the dispersion of small particles in a simplified turbulent flow field

"APPLICATIONS" : 1. Plankton's anomalous rise velocity

2. Vertical Mixing & Transport of nutrients and fine sediments in upper ocean due to wind driven Langmuir circulations (*)



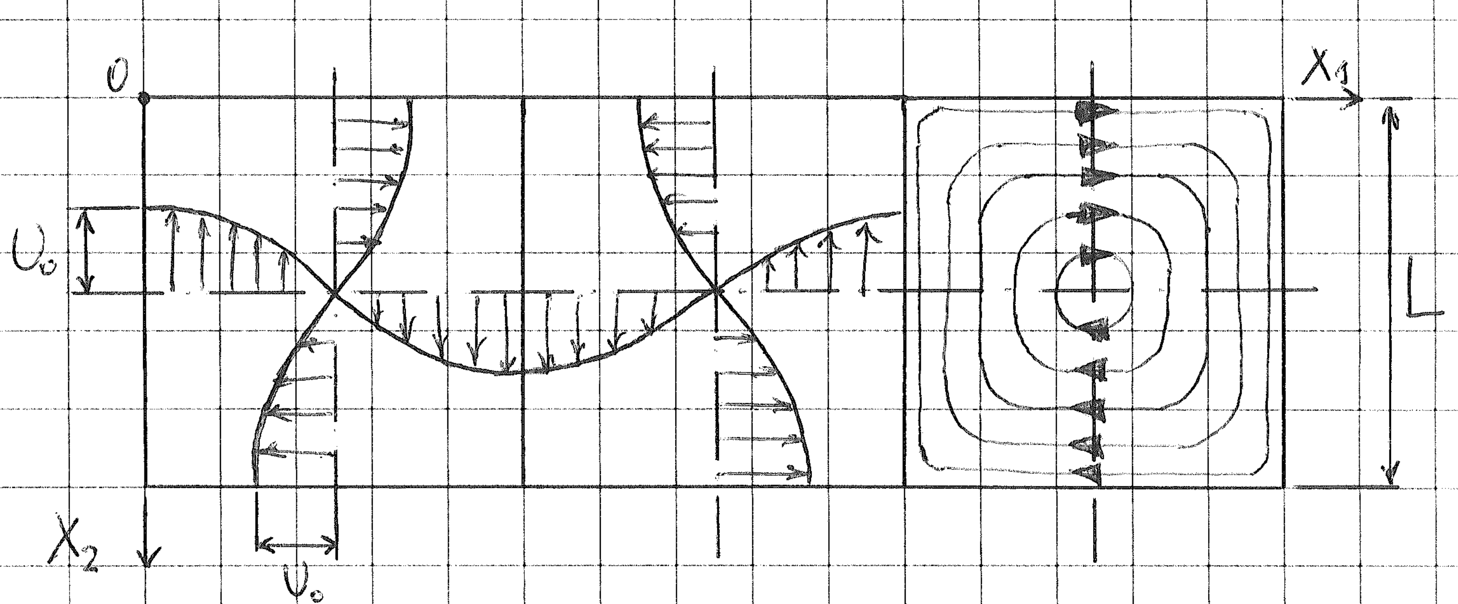
(*) Langmuir circulations are probably due to inviscid instability generated by non linear interaction between SURFACE GRAVITY WAVES and WIND-INDUCED SHEAR

(A) FLOW FIELD

2D periodic flow of cellular vortices

Incompressible

$$\psi = U_0 L \sin(X_1/L)\sin(X_2/L)$$



Fluid velocity components:

$$u_1 = \frac{dX_1}{dt} = U_0 \sin(X_1/L)\cos(X_2/L) = \frac{\partial \psi}{\partial X_2}$$

$$u_2 = \frac{dX_2}{dt} = -U_0 \cos(X_1/L)\sin(X_2/L) = -\frac{\partial \psi}{\partial X_1}$$

(B) PARTICLE MOTION EQUATIONS

Start from Maxey & Riley (1983):

$$\underbrace{m_p \frac{d\vec{V}}{dt}}_{\text{INERTIA}} = \underbrace{(m_p - m_f)\vec{g}}_{\text{BUOYANCY}} + \underbrace{m_f \left.\frac{D\vec{u}}{Dt}\right|_{\vec{Y}(t)}}_{\text{EFFECTS OF PRESSURE GRAD. OF THE UNDISTURBED FLOW}}$$

$$\frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}$$

$$\frac{d\vec{u}}{dt} = \frac{\partial \vec{u}}{\partial t} + \vec{v}\cdot\nabla\vec{u}$$

$$\underbrace{-\frac{1}{2} m_f \frac{d}{dt}\left[\vec{V} - \vec{u}\big|_{\vec{Y}(t)}\right]}_{\text{ADDED MASS}} \underbrace{- 6\pi a \mu \left[\vec{V} - \vec{u}\big|_{\vec{Y}(t)}\right]}_{\text{STOKES DRAG}}$$

ADDED MASS { FORZA RICHIESTA PER ACCELERARE IL FLUIDO INTORNO ALLA PARTICELLA

$$\vec{X} \triangleq \left[\vec{V} - \vec{u}\big|_{\vec{Y}(t)}\right]$$

$$\underbrace{- 6\pi a^2 \mu \int_0^t \frac{d\vec{X}/d\tau}{\sqrt{\pi\nu(t-\tau)}}\, d\tau}_{\text{BASSET HISTORY TERM}} + \underbrace{6{,}46\, \rho_f\, a^2 \sqrt{\nu}\cdot\Gamma}_{\text{LIFT}}$$

$$m_f C_L \frac{(\vec{u} - \vec{V}) \times (\nabla \times \vec{u})}{\sqrt{|\nabla \times \vec{u}|}}$$

$$m_p = \rho_p \cdot \frac{4}{3}\pi a^3 \qquad \text{MASS OF THE PARTICLE}$$

$\hookrightarrow a =$ PARTICLE RADIUS

$$m_f = \rho_f \cdot \frac{4}{3}\pi a^3 \qquad \text{MASS OF FLUID DISPLACED BY THE PARTICLE}$$

$$\vec{V} \triangleq \vec{V}(t) \triangleq V_1\vec{i} + V_2\vec{j} \qquad \text{PARTICLE VELOCITY}$$

$$\vec{u} \triangleq u_1\vec{i} + u_2\vec{j} \qquad \text{FLUID VELOCITY @ PART. POS}$$

$$\vec{Y}(t) \triangleq Y_1(t)\vec{i} + Y_2(t)\vec{j} \qquad \text{PARTICLE POSITION}$$

$$\Gamma \triangleq \frac{\left[(\vec{u}-\vec{V})\times(\vec{\nabla}\times\vec{u})\right]}{\sqrt{|\vec{\nabla}\times\vec{u}|}} \qquad \text{with } \vec{\nabla}\times\vec{u} \triangleq \text{VORTICITY}$$

Neglecting Basset and re-arranging:

$$\frac{d\vec{V}}{dt} = \overset{\vec{W}^{(1)}\alpha}{\frac{m_p - m_f}{m_p + \frac{1}{2}m_f}}\,\vec{g} + \frac{m_f}{m_p + \frac{1}{2}m_f}\,\vec{u}\cdot\vec{\nabla}\vec{u}$$

$$+ \frac{1}{2}\overset{R}{\frac{m_f}{m_p + \frac{1}{2}m_f}}\,\vec{V}\cdot\vec{\nabla}\vec{u} - \overset{\alpha = 1/\tau_p}{\frac{6\pi a\mu}{m_p + \frac{1}{2}m_f}}(\vec{V}-\vec{u})$$

$$+ \frac{6{,}46\,\rho_f a^2\sqrt{\nu}}{m_p + \frac{1}{2}m_f}\cdot\Gamma$$

$$\left[\alpha = \frac{6\pi a\mu}{\frac{4}{3}\pi a^3(\rho_p + \frac{1}{2}\rho_f)} = \frac{9}{2}\frac{\mu}{(\rho_p + \frac{1}{2}\rho_f)a^2} = \frac{1}{\tau_p}\right]$$

NB nel flusso cellulare la der. euleriana $\partial \vec{u}/\partial t = 0$ "

Define :

use R, A, W $\Longleftrightarrow$ * $0 < R \leq 0{,}4$ AEROSOL RANGE

* $0{,}4 < R < 2$ TRANSITION RANGE

use R, B, Q $\Longleftrightarrow$ * $R = 2$ BUBBLE RANGE

- $$R \triangleq \frac{m_f}{m_p + \frac{1}{2} m_f}$$ MASS RATIO

$$\Longrightarrow \text{if } \rho_p \gg \rho_f \text{ then } R \to 0$$

$$\Longrightarrow \text{if } \rho_p \ll \rho_f \text{ then } R \to 2$$

- $$\alpha \triangleq \frac{6 \pi a \mu}{m_p + \frac{1}{2} m_f}$$ INERTIA PARAMETER

$$\tau_p = \frac{(\rho_p + \frac{1}{2}\rho_f) D_p^2}{18 \mu} = \frac{(\rho_p + \frac{1}{2}\rho_f)}{18\mu} \frac{\pi D_p^3}{6} \cdot \frac{6}{\pi D_p} = \frac{m_p + \frac{1}{2} m_f}{3 \pi \mu D_p} = \frac{m_p + \frac{1}{2} m_f}{6 \pi a \mu} = 1/\alpha \text{ !!}$$

- $$\vec{W}^{(s)} \triangleq \frac{m_p - m_f}{6 \pi a \mu} \vec{g}$$ STOKES SETTLING VEL.

$$\Rightarrow \vec{W}^{(s)} = \frac{m_p - m_f}{6 \pi a \mu} \vec{g} \cdot \frac{m_p + \frac{1}{2} m_f}{m_p + \frac{1}{2} m_f}$$

$$= \frac{m_p - m_f}{m_p + \frac{1}{2} m_f} \vec{g} \cdot \frac{1}{\alpha}$$

- $$\mathcal{L} \triangleq \frac{6{,}46 \rho_f a^2 \sqrt{\nu}}{m_p + \frac{1}{2} m_f} = \ldots = \frac{1{,}5422}{a} \sqrt{\frac{\mu}{\rho_f}} \cdot R$$

to get:

$$\frac{d\vec{V}}{dt} = \alpha \vec{W}^{(s)} + R\left(\vec{u}\cdot\vec{\nabla}\vec{u} - \frac{1}{2}\vec{V}\cdot\nabla\vec{u}\right) - \alpha(\vec{V} - \vec{u}) + \mathcal{L}\,\Gamma \qquad (1)$$

Equation in non-dimensional form:

take $\vec{V}^* = \vec{V}/U_0$

$\vec{u}^* = \vec{u}/U_0$

$\vec{W}^{(s)*} = \vec{W}^s/U_0$

$\vec{X}^* = \vec{X}/L$ $\qquad + \quad A = \dfrac{\alpha L}{U_0}$

$\vec{Y}^* = \vec{Y}/L$

$t^* = t\,.\,U_0/L$

substitute into (1) and re-arrange to obtain:

$$\frac{d\vec{V}^*}{dt^*} = A\left[\vec{W}^{(s)*} + (\vec{u}^* - \vec{V}^*)\right] + R\left[\vec{u}^*\cdot\vec{\nabla}^*\vec{u}^* + \frac{1}{2}\vec{V}^*\,\vec{\nabla}\vec{u}^*\right]$$

$$+ 0{,}727\sqrt{A\cdot R}\;.\;\frac{\left[(\vec{u}^* - \vec{V}^*)\times(\vec{\nabla}^*\times\vec{u}^*)\right]}{\sqrt{|\vec{\nabla}^*\times\vec{u}^*|}} \qquad (2)$$

Dropping apex *, Eq. (2) in scalar form reads as:

$$\frac{dV_i}{dt} = A(u_i - V_i) + R\left[u_j \frac{\partial u_i}{\partial x_j} + \frac{1}{2} V_j \frac{\partial u_i}{\partial x_j} \right] +$$

$$+ A W_i^{(s)} + \frac{0{,}727 \sqrt{AR}\, \left[(\vec{u} - \vec{V}) \times (\vec{\nabla} \times \vec{u}) \right] \Big|_i}{\sqrt{|\vec{\nabla} \times \vec{u}|}\, \Big|_i}$$

where $i = 1, 2$ and $j = 1, 2$.

Note:

$$(\vec{u} - \vec{V}) \times \underbrace{(\vec{\nabla} \times \vec{u})}_{\text{VETT.}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 - V_1 & u_2 - V_2 & 0 \\ 0 & 0 & \frac{\partial u_2}{\partial y_1} - \frac{\partial u_1}{\partial y_2} \end{vmatrix} =$$

$$\frac{\partial u_2}{\partial x_1} = \frac{\partial}{\partial x_1}\left[-U_0 \cos\left(\frac{x_1}{L}\right) \sin\left(\frac{x_2}{L}\right) \right] = \frac{U_0}{L} \sin\left(\frac{x_1}{L}\right) \sin\left(\frac{x_2}{L}\right)$$

$$\frac{\partial u_2}{\partial y_1} = \frac{U_0}{L} \sin\left(\frac{y_1}{L}\right) \sin\left(\frac{y_2}{L}\right) \quad \text{DIM.}$$

$$\frac{\partial u_2/U_0}{\partial y_1/L} = \frac{\partial u_2^*}{\partial y_1^*} = \sin(y_1^*) \sin(y_2^*) \quad \text{ADIM.}$$

ETC. ETC.

$$= \vec{i} \left[(u_2 - V_2)\left(\frac{\partial u_2}{\partial y_1} - \frac{\partial u_1}{\partial y_2} \right) \right]$$

$$- \vec{j} \left[(u_1 - V_1)\left(\frac{\partial u_2}{\partial y_1} - \frac{\partial u_1}{\partial y_2} \right) \right] =$$

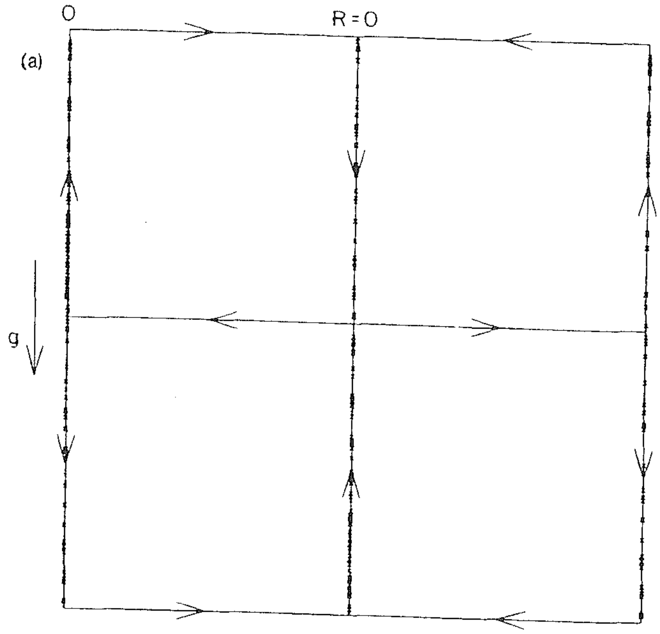
$$= \vec{i} \left[2 \sin Y_1 \sin Y_2 \left(-\cos Y_1 \sin Y_2 - V_2 \right) \right] + \vec{j} \left[2 \sin Y_1 \sin Y_2 \left(V_1 - \sin Y_1 \cdot \cos Y_2 \right) \right]$$

In summary, the equations to be integrated in time for tracking particles in the cellular flow are: $\boxed{7}$

$$\frac{1}{A}\frac{dV_1}{dt} + V_1 = \sin Y_1 \cos Y_2 + \frac{1}{2}\frac{R}{A}\left(V_1 \cos Y_1 \cos Y_2 - V_2 \sin Y_1 \sin Y_2\right) + \frac{R}{A} \sin Y_1 \cos Y_1 + 0{,}727\sqrt{\frac{R}{A}} \cdot \frac{(V_2 - \cos Y_1 \sin Y_2)(+2 \sin Y_1 \sin Y_2)}{\sqrt{2\left|\sin Y_1 \sin Y_2\right|}}$$

$$\frac{1}{A}\frac{dV_2}{dt} + V_2 = -\cos Y_1 \sin Y_2 + \frac{1}{2}\frac{R}{A}\left(V_1 \sin Y_1 \sin Y_2 - V_2 \cos Y_1 \cos Y_2\right) + \frac{R}{A} \sin Y_2 \cos Y_2 + W^{(s)} + 0{,}727\sqrt{\frac{R}{A}} \cdot \frac{(V_1 - \sin Y_1 \cos Y_2)\, 2 \sin Y_1 \sin Y_2}{\sqrt{2\left|\sin Y_1 \sin Y_2\right|}}$$

$$\underset{(2)}{V_1} \Rightarrow \frac{dY_1}{dt} = \underset{(2)}{V_1} \Rightarrow \underset{(2)}{Y_1^{n+1}} = \underset{(2)}{Y_1^{n}} + \underset{(2)}{V_1} \cdot dt$$



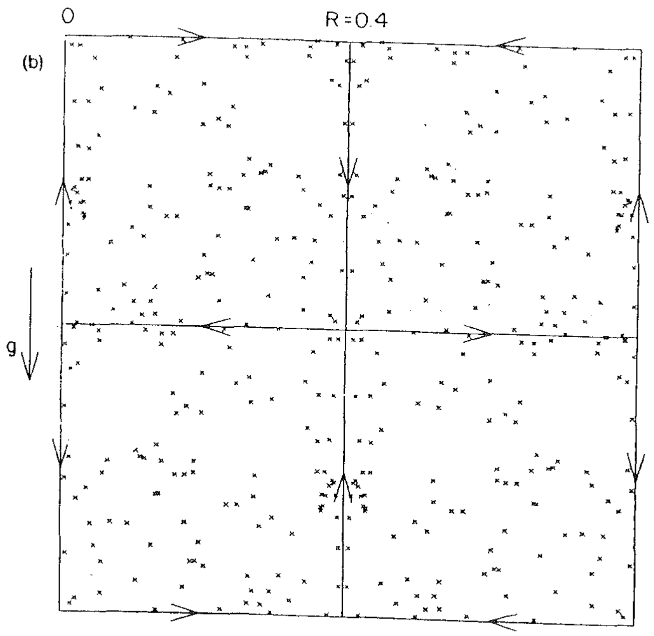


FIG. 10. Particle position plots at $t = 80$ for particles with inertia parameter $A = 2$, still-fluid settling velocity $W = 2.0$: (a) mass ratio parameter $R = 0$; (b) 0.4.

W greater than about 1 the particles collect along the vertical boundaries and the settling velocity $\langle V_2 \rangle$ is reduced by the upflow regions. An equilibrium point exists for $W = 1.02$, and while it is unstable, the growth rates are so small that it is close to being neutrally stable.

The general conclusions from these and other computations are first, that for particles significantly denser than the surrounding fluid, particle suspension does not occur. Second, the asymptotic merging of trajectories into isolated paths is a general feature that is not sensitive to the value of R within this range. Finally, the average settling velocity of the particles is greater than that in still fluid if the still-fluid settling velocity W is small, but is significantly reduced when the value of W approaches 1.

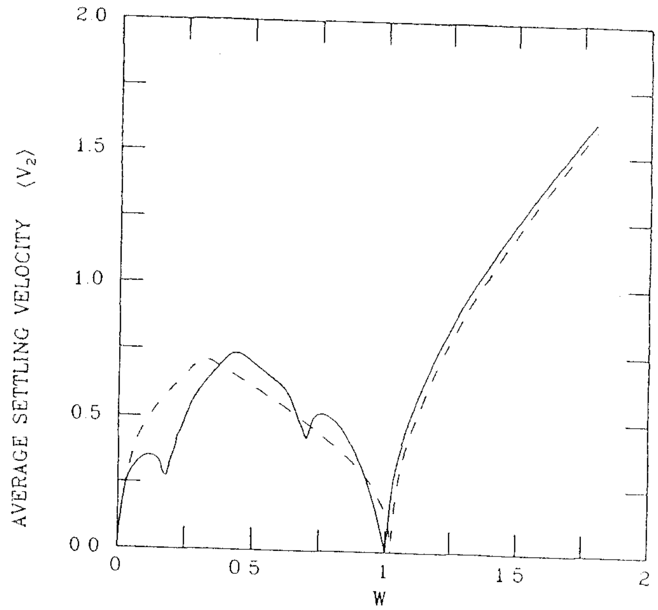


FIG. 11. Average particle settling velocity $\langle V_2 \rangle$ against still-fluid settling velocity W for $A = 2$: —, $R = 0.0$; ----, $R = 0.4$.

V. BUBBLE LIMIT: $R = 2$

The stability analysis in Sec. III indicates that for bubble particles stable equilibrium points exist which may lead to the permanent suspension of the particles. To test this possibility, some sample trajectories have been computed for bubble particles $R = 2$, and for a still-fluid, bubble rise speed Q of 0.5; see Fig. 12(a). The inertia parameter of the bubble B is set equal to 10.0, the same as for the stability diagram in Fig. 4. Six of the particles released spiral in toward interior equilibrium points and never escape from their initial cell, while a seventh particle is trapped after passing through two cells. The two particles that do rise through the cells zigzag through the upflow regions of each cell, rising faster than in still fluid. In contrast, Fig. 12(b) shows the particle trajectories for $Q = 1.25$. Here there are no equlibrium points and all the particles escape, rising rapidly through the upflow regions of each cell.

The particle position diagram for $Q = 0.5$, Fig. 13, shows the corresponding positions of an initially uniform array of particles at $t = 20$, the time at which the computations in Fig. 12 were stopped. There is a strong clustering of particles about the equilibrium points, and if the computations are continued further it is found that 90% of the particles spiral in toward these points and are trapped. The particle position plots can also be used to find the long-term asymptotic form of the particle motion and whether or not bubble particles exhibit the same merging of trajectories found for aerosol particles. The long-term behavior for $Q = 1.25$ is shown in Fig. 14(a). All the particles here have collected along simple, well-defined curves that pass through the central region of each cell and are biased toward the upflow region of each cell. The merging of trajectories into isolated paths thus still occurs. But the accumulation for rapidly rising bubbles is between vertical cell boundaries, as opposed to on the boundaries as found for rapidly settling aerosol particles. Figure 14(b) gives the asymptotic behav-

Possible projects :

1. Compute average settling/rising velocity $\langle V_z \rangle$ and compare against Stokes velocity $W^{(s)}$

2. Calculate equilibrium points and perform linear stability analysis ⟹

3. Compute the compressibility of the particle phase ⟶ Jacobian

Do this considering different values of the particle-to-fluid density ratio ρ_p/ρ_f and different sets of forces acting on the particles

(I : M&R87 ; II : M&R87 + Added Mass ;
III : M&R87 + Lift + Added Mass)

Technical details:

- INITIAL & BOUNDARY CONDITION

⇒ Initial particle position : random distr.

⇒ Initial particle velocity :

× equal to zero ($V_1 = V_2 = 0$ @ $t=0$)

× equal to the fluid velocity at the particle initial position

($V_i = u_i|_{Y_i(t)}$ @ $t=0$)

⇒ Boundary condition : periodic

- TIME INTEGRATION

⇒ Runge-Kutta (4^{th}-order)

× subroutine available at

www.nr.com

$\Longrightarrow$ Solve for the following system of 1^{st}-order ODEs :

$$
\begin{cases}
\dfrac{dY_1(t)}{dt} = V_1(t) \\[2ex]
\dfrac{dY_2(t)}{dt} = V_2(t) \\[2ex]
\dfrac{dV_1(t)}{dt} = A \cdot \left[-V_1 + \sin Y_1 \cos Y_2 + \dots \right] \\[2ex]
\dfrac{dV_2(t)}{dt} = A \cdot \left[-V_2 - \cos Y_1 \sin Y_2 + \dots \right]
\end{cases}
$$