Vorticity and Dynamics
In Navier-Stokes equation

Nonlinear term

$$\omega \times \vec{u} \quad \text{the Lamb vector is related to the nonlinear term}$$

$$(\vec{u} \cdot \nabla) \vec{u} = \nabla \left( \frac{\vec{u}^2}{2} \right) + \omega \times \vec{u}$$

Sort of Coriolis force in a rotation frame

Viscous term

$$\nu \Delta \vec{u} = \nu \left( \nabla (\nabla \cdot \vec{u}) - \nabla \times (\nabla \times \vec{u}) \right) = -\nu \nabla \times \omega$$
\[ \frac{\partial \vec{u}}{\partial t} + \vec{\omega} \times \vec{u} + \nabla \left( \frac{\vec{u}^2}{2} \right) = -\nabla \frac{p}{\rho} - \nu \nabla \times \vec{\omega} \]

taking the rotational

\[ \frac{\partial \vec{\omega}}{\partial t} + \nabla \times (\vec{\omega} \times \vec{u}) = -\nu \Delta \vec{\omega} \]

Dynamical equation for Vorticity
Pressure is eliminated in the equation

\[ \Delta p = 2 \rho Q \quad Q = \frac{1}{2} \left[ \frac{1}{2} \omega^2 - e^2 \right] \]

\( Q > 0 \)  \textbf{Q-Criterium to define a vortex region}
Mixing layer

\[ \ddot{u} = u_x(y) \dot{e}_x; \quad \omega = -\frac{\partial u}{\partial y} \]

\[ e_{yx} = e_{xy} = \frac{1}{2} \frac{\partial u}{\partial y} \]

\[ e_{xx} = e_{yy} = 0 \]

\[ \Rightarrow \quad \omega^2 = 2 e_{ij} e_{ij} \quad Q = 0 \]

\[ \Rightarrow \quad \text{pressure is uniform} \]

Vortex

\[ \ddot{u} = u_\theta(r) \dot{e}_\theta; \quad \omega_z = \frac{1}{r} \frac{\partial (ru_\theta)}{\partial r} \]

\[ e_{rr} = 0, \quad e_{\theta\theta} = 0, \quad e_{r\theta} = r \frac{\partial}{\partial r} \left( \frac{u_\theta}{r} \right) \]
\[ \Delta p = 2 \rho Q \quad Q = \frac{1}{2r} \frac{\partial u_\theta^2}{\partial r} \]

Pressure can be computed using the first component of the Navier-Stokes equation in polar coordinates

\[ \frac{u_\theta^2}{r} = \frac{1}{\rho} \frac{\partial p}{\partial r} \quad p(r) = p(\infty) - \rho \int_r^\infty \frac{u_\theta^2}{r} dr \]

pressure decreases as we get near the vortex center: it is a pressure low.
Q-vortex from Delcayre (2000)

Figure 6. Threshold effects on near-wall vortices shown by the Q selection:
(a) $Q = 0.2$, (b) $Q = 0.4$, (c) $Q = 0.6$, (d) $Q = 0.8$, (e) $Q = 1.0$ and (f) $Q = 1.2$. 
Cavitation (Cadot et al)
Dissipation and Vorticity

Internal dissipation $\Phi$ per unit volume

particle deformation $\rightarrow$ internal energy

$\Phi = 2 \mu e_{ij} e_{ij} > 0$ \quad incompressibility assumed

$\Phi = \mu \omega^2 + 2\mu \nabla \cdot [\omega \times \vec{u} + 12 \nabla (\vec{u}^2)]$
Dissipation and Vorticity

\[
\int_v \Phi \, d\tau = \mu \int_v \omega^2 \, d\tau + \mu \int_{\text{surface}} [2 \, \omega \times \vec{u} + \nabla (\vec{u}^2)] \cdot \vec{n} \, dS
\]

the dissipation per unit mass used in turbulence

\[
\epsilon = \frac{1}{\rho V} \int_v \Phi \, d\tau = \frac{\nu}{V} \int_v \omega^2 \, d\tau
\]

There exists of a zone of dissipation around a vortex
Chaoticity and Vorticity

\[ \vec{u} = \vec{u}_{BS} + \nabla \phi \]

Potential only depends on the boundary conditions since it satisfies at each time

\[ \Delta \phi = 0 \quad \vec{n} \cdot \nabla \phi = (\vec{u} - \vec{u}_{BS}) \cdot \vec{n} \]

Past history does not play a role for \( \phi \)

Vorticity precisely brings this historical aspect.

Vorticity \( \omega \) modifies velocity \( \vec{u} \)

Velocity \( \vec{u} \) modifies with vorticity \( \omega \)
This implies feedback loop

⇒ Chaoticity and vortex

⇒ Turbulence and vortex
Transport Equation for Vorticity:
Incompressible Newtonian Fluid

$$\nabla \times [\omega \times \vec{u}] = (u \cdot \nabla)\vec{\omega} - (\omega \cdot \nabla)\vec{u}$$

$$\frac{D\omega}{Dt} = [\partial_t + \vec{u} \cdot \nabla] \omega = \omega \cdot \nabla \vec{u} + \nu \Delta \omega$$

Circulation theorem
$$\frac{d\Gamma}{dt} = -\nu \oint_C \nabla \times \omega \cdot d\vec{l}$$
Vorticity Diffusion

\[ \begin{align*}
    U_\theta(r, t) &= \frac{\Gamma}{2\pi r} \left[ 1 - \exp\left( -\frac{r^2}{a^2} \right) \right] \\
    \Omega(r, t) &= \frac{\Gamma}{\pi a^2} \exp\left( -\frac{r^2}{a^2} \right) \\
    a^2(t) &= a^2(0) + 4\nu t
\end{align*} \]

Lamb-Oseen Vortex
Consider a vector element of a material line \( \vec{AB} = \delta \vec{l} \)

During time \( dt \), points A and B move respectively to points A' and B'

\[
\vec{A'B'} = \delta \vec{l}' = \delta \vec{l} + (\vec{u}_B - \vec{u}_A)\delta t
\]

\[
\frac{D\delta \vec{l}}{Dt} = \frac{\delta \vec{l} - \delta \vec{l}'}{\delta t} \quad \text{and} \quad \vec{u}_B - \vec{u}_A = (\vec{AB} \cdot \vec{u})
\]

\[
\frac{D\delta \vec{l}'}{Dt} = (\delta \vec{l} \cdot \nabla)\vec{u}
\]
Inviscid incompressible flows

By comparing equations

\[
\frac{D\omega}{Dt} = \omega \cdot \nabla u \quad \frac{D\delta \vec{l}}{Dt} = (\delta \vec{l} \cdot \nabla) \vec{u}
\]

In the inviscid fluid, for a given fluid particle, vorticity evolves in time the same way as does a small material line with the same direction.

Concept used in numerical vortex methods.
Vorticity Enhancement by Strain Field

\[ \frac{D\omega}{Dt} = [\partial_t + u \nabla] \omega = \omega \nabla u + \nu \Delta \omega \]

Stretching
Law of conservation:
Helmholtz Laws for inviscid incompressible flows
First Law of conservation

Circulation around a material loop

\[ \Gamma = \oint_{C} \mathbf{u} \cdot d\mathbf{l} = \int_{S} \mathbf{\omega} \cdot d\mathbf{S} \]

\[ \frac{d}{dt} \int_{C(t)} \mathbf{u} \cdot d\mathbf{x} = \int_{C(t)} \frac{D\mathbf{u}}{Dt} \cdot d\mathbf{x} \]

\[ \frac{d}{dt} \Gamma = -\nu \int_{C(t)} \nabla \times \mathbf{\omega} \cdot d\mathbf{x} \quad \nu = 0 \quad \frac{d}{dt} \Gamma = 0 \]

Circulation along a material line remains constant when convected by the fluid.
Second Law of conservation

Consider a fluid particle without vorticity

All small loop around the particle would have zero circulation

\[ \frac{d}{dt} \Gamma = 0 \quad \Rightarrow \quad \text{all small loop remain of zero circulation} \]

\[ \omega = 0 \]

A potential region remains potential when convected by the fluid in the inviscid context
Third Law of conservation.

Vortex lines are material lines i.e. convected by the fluid

↓

Tilting and Stretching of Vortex lines

First and Third laws imply that:
Circulation of a vortex tube does not change with time
Helmholtz conservation laws in 2D

For the two-dimensional case \( u(x,y), \ v(x,y), w=0 \)

\[
\omega = \omega \mathbf{e}_z = \left( \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \right) \mathbf{e}_z
\]

Vorticity lines are straight and there is no stretching

\[
(\bar{u} \cdot \nabla) \omega = 0
\]

In 2D Euler, vorticity is such that

\[
\frac{D\omega}{Dt} = 0
\]

\[\Rightarrow\] Vorticity of each particle is hence conserved
Helmholtz conservation laws in axisymmetric case

axisymmetric vortex ring

\[ \bar{u} = u_r(r, z, t) \hat{e}_r + u_z(r, z, t) \hat{e}_z \]
\[ \omega = \omega_\theta(r, z) \hat{e}_\theta \]
\[ \omega_\theta = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \]

Vorticity lines are rings and there is stretching

In Euler, vorticity is such that

\[ \frac{D}{Dt} \left( \frac{\omega_\theta}{r} \right) = 0 \]

\[ \Rightarrow \frac{\omega_\theta}{r} \quad \text{is hence conserved} \]