VORTEX IN HYDRODYNAMICS : DEFINITIONS

Vorticity is defined as

$$\mathbf{\Omega}(\mathbf{x}, t) = \nabla \times \mathbf{U}$$

$$\omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}$$
$$\omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$
$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\omega_r = \frac{1}{r}\,\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}$$
$$\omega_\theta = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}$$
$$\omega_z = \frac{1}{r}\,\frac{\partial (r\,u_\theta)}{\partial r} - \frac{1}{r}\,\frac{\partial u_r}{\partial r}$$

Vorticity is measured in SI units: 1/sec

Vortex versus Vorticity

Vortex is a Coherent Flow Structure :
a geometrical or practical object difficult to formalize

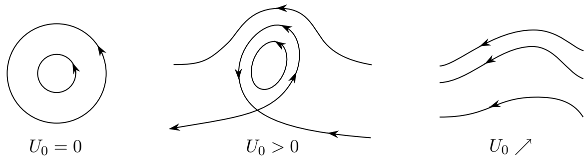
Vorticity ω is a mathematical concept.

Spatial coherence $\longrightarrow$ Compact field

Time coherence $\longrightarrow$ Conservation Law

Why one choses vorticity to identify structures

Dependance upon the chosen referential of the observer of patterns based on streamlines



Versus

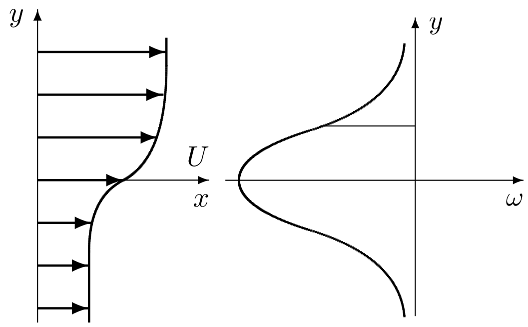
Galilean invariance of patterns based on vorticity

Vorticity Solutions with symmetry

The Solid body rotation

$u = \frac{1}{2}\omega \times \vec{r}$ an exact Navier-Stokes solution in which vorticity is a constant.

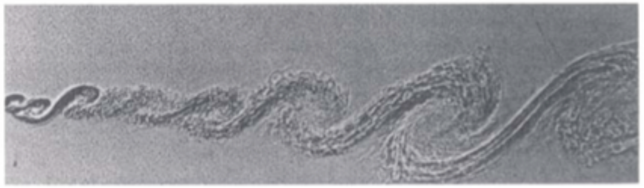
The plane mixing layer



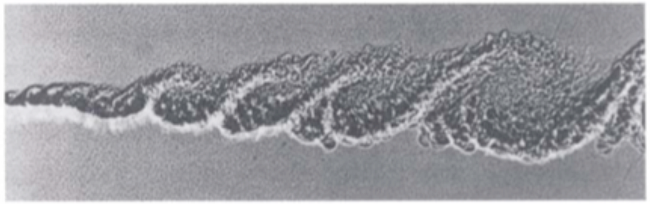
$$\vec{u} = U(y)\,\vec{e}_x$$

$$\omega = \left(-\frac{\partial U}{\partial y}\right)\vec{e}_z$$

Kelvin-Helmholtz Instability



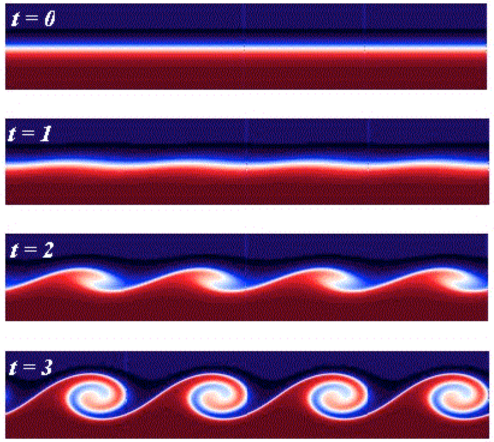
(a)



(b)

Fig. 2.7. Large scale structures in mixing layers at high Reynolds numbers. Re in (b) is doubled with respect to Re in (a). After Van Dyke (1982).

Kelvin-Helmholtz Instability



Temporal problem : DNS

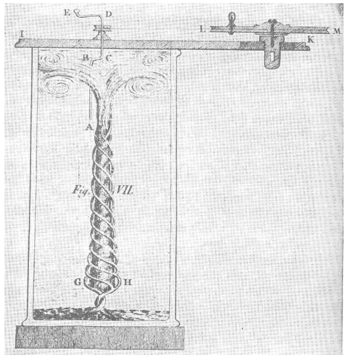
Inflexion point for velocity field Maximum of $|\omega|$

Shear Instability

A Columnar Vortex

$$u_\theta(r) \quad u_z(r) \quad u_r = 0 \quad \Rightarrow \quad \boxed{\omega_\theta = -\frac{\partial u_z}{\partial r} \qquad \omega_z = \frac{1}{r}\frac{\partial(r\,u_\theta)}{\partial r}}$$

Vortex + Jet

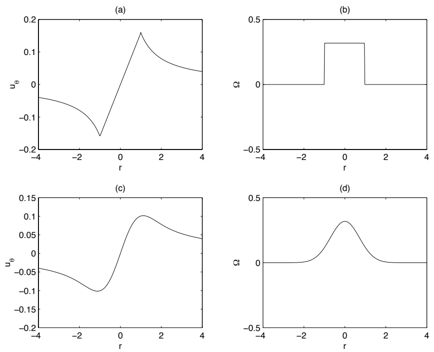


Wilcke, Stockholm 1780

Tip vortices

Rankine Vortex

Lamb-Oseen Vortex



Rankine Vortex

$$U_\theta(r,t) = \frac{\Gamma}{2\pi a^2} r \ , \Omega(r,t) = \frac{\Gamma}{\pi a^2} \ , r < a$$

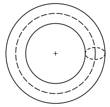
$$U_\theta(r,t) = \frac{\Gamma}{2\pi r} \ , \Omega(r,t) = 0 \ , r > a$$

Lamb-Oseen Vortex

$$U_\theta(r,t) = \frac{\Gamma}{2\pi r}[1 - \exp(-\frac{r^2}{a^2})]$$

$$\Omega(r,t) = \frac{\Gamma}{\pi a^2} \exp(-\frac{r^2}{a^2})$$

A Vortex Ring

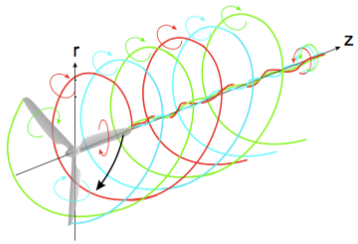


$$\vec{u} = u_r(r,z,t)\,\vec{e}_r + u_z(r,z,t)\,\vec{e}_z$$

$$\omega = \omega_\theta(r,z,t)\,\vec{e}_\theta \qquad \omega_\theta(r,z,t) = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}$$

Vortex due to a Volcano

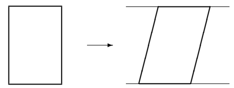
A helical Vortex



Kinematical interpretation of Vorticity

Following a Fluid Particle

$$\vec{u} = U(y)\,\vec{e}_x$$



Global Displacement

Rotation

Deformation

It is a measure of the rate of rotation of a fluid element

The relative fluid motion nearby a point A moving with the fluid particle.

Let B be a point in the vicinity of A.

$$\left(\vec{u}(B) - \vec{u}(A)\right)_i \equiv \delta u_i = \left(\frac{\partial u_i}{\partial x_j}\right) \delta x_j$$ Taylor expansion

$$\delta u_i = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\equiv e_{ij}} \delta x_j + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\equiv \Omega_{ij}} \delta x_j$$ Decomposition

$$\delta u_i = e_{ij}\, \delta x_j + \Omega_{ij}\, \delta x_j$$

e_{ij} Deformation rate tensor (symmetric)

Ω_{ij} Rotation tensor (antisymmetric)

$$\Omega_{ij}\,\delta x_j = \left(\left(\frac{\omega}{2}\right)\times\delta x\right)_i \qquad \omega_k = \epsilon_{kmn}\partial_m U_n$$

$$\delta u_i = e_{ij}\,\delta x_j + \left(\left(\frac{\omega}{2}\right)\times\delta x\right)_i$$

The second term expresses a solid body rotation along an axis parallel to the vorticity vector.

$$e_{ij} = \frac{1}{2}(\partial_i U_j + \partial_j U_i)$$ Symmetric Tensor

$$\frac{D}{Dt}\left((d\vec{x})^2\right) = 2\,(d\vec{x})\cdot\frac{D}{Dt}(d\vec{x}) = 2\,d\vec{x}\cdot\vec{e}\cdot d\vec{x}$$ expresses the deformation

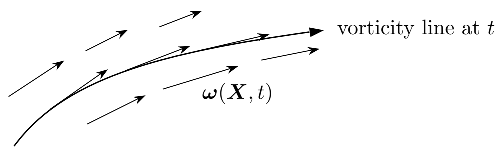
Dynamical interpretation of Vorticity

If the fluid particle possesses spherical symmetry,
Its angular momentum in the barycentric referential frame is

$$(\delta\vec{\sigma})_i = I\,\omega_i$$

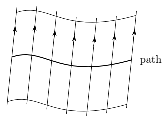
$I_{jl} = I\,\delta_{jl}$ is the inertia tensor

Vorticity lines



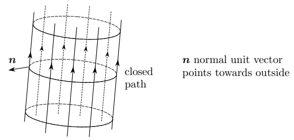
Oriented lines whose tangents are in the same direction as $\vec{\omega}(\vec{X})$

Vorticity surfaces



A set of vorticity lines passing through a given path

Vorticity tube



path is closed

Topology of Vorticity Lines

$$\vec{u} = \begin{pmatrix} A\sin[\alpha z] + C\cos[\alpha y] \\ B\sin[\alpha x] + A\cos[\alpha z] \\ C\sin[\alpha y] + B\cos[\alpha x] \end{pmatrix}$$

ABC Flow:

Steady Euler solution

This a Beltrami flow $\vec{\omega} = \alpha\vec{u}$

Vorticity lines= streamlines = Particle trajectories

They follows the ODE equation $\frac{d\vec{x}}{dt} = \vec{u}(\vec{x})$

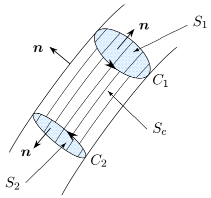
$A = B = 0$ Vorticity lines are straight lines

For some values of A, B, C, the system is chaotic

Vorticity lines can have a very complicated pattern

Vortex tubes: Kinematics

Vorticity is a divergenceless quantity $$\frac{\partial \omega_i}{\partial x_i} = 0$$



$$0 = \iint_{S_1} \vec{\omega} \cdot \vec{n}\, dS + \iint_{S_2} \vec{\omega} \cdot \vec{n}\, dS + \iint_{S_e} \vec{\omega} \cdot \vec{n}\, dS$$

$$\Gamma \equiv \oint_{C_1} \vec{U} \cdot d\vec{X} = \iint_{S_1} \vec{\omega} \cdot \vec{n}\, dS = \oint_{C_2} \vec{U} \cdot d\vec{X} = -\iint_{S_2} \vec{\omega} \cdot \vec{n}\, dS$$

Characteristic of the vortex tube: its circulation

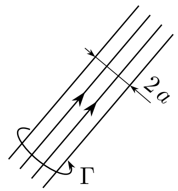
Vorticity tube surrounded by an irrotational fluid: a vortex

Γ circulation giving its strength or intensity.

$[\Gamma] = L^2 T^{-1}$ $\quad \mathrm{m}^2.\mathrm{s}^{-1}$

$Re = \dfrac{\Gamma}{\nu}$ Reynolds of a Vortex

a core size



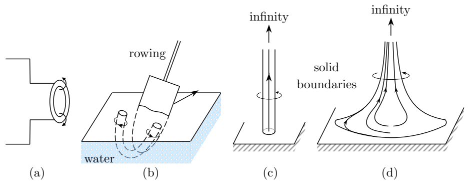
Topology of Vortex tubes

Vorticity tubes are either

A) closed,

B) go to infinity,

C) end on surface boundaries

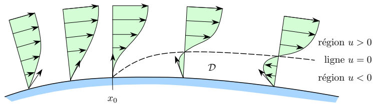


Walls and Vorticity

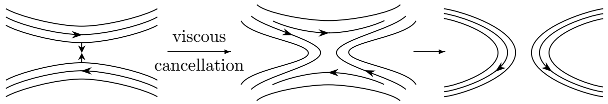
On no-slip Boundaries

$$\oint_{C_s} \vec{U} \cdot d\vec{X} = 0 \qquad \forall C_s \qquad \vec{\omega} \,//\, \text{ to a no-slip surface}$$

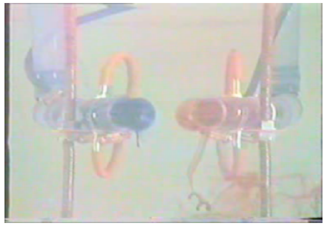
Vorticity at a separation point on a no-slip wall



$$\omega_z = -\frac{\partial u}{\partial y} = 0$$



Topological changes : reconnections



From Lim's website