Vorlesung 11

- Reynolds's stress. Modelling & Law of the Wall

Following the Reynolds' procedure, we determined the following form for the average Navier Stokes equations:

$$\rho\left(\overline{v}_j \frac{\partial \overline{v}_i}{\partial x_j}\right) = -\frac{\partial \overline{P}}{\partial x_i} + \mu \frac{\partial^2 \overline{v}_i}{\partial x_j^2} - \rho \frac{\partial \overline{v_i' v_j'}}{\partial x_j}$$

$$= -\frac{\partial \overline{P}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial \overline{v}_i}{\partial x_j} - \rho \overline{v_i' v_j'}\right)$$

ρ and μ are uniform

In Vorlesung 10 we examined the physical meaning of the Reynolds' stresses and we concluded that they are always (probably!) negative so that they contribute with an extra drag.

The analysis of a fluid parcel made in Vorlesung 10 is much like the ~~analysis~~ analysis of the molecular motion done by Boussinesq (1877) suggesting a possible way to model the Reynolds' stresses with a "model" viscosity :

$$\tau_{yx}^{(t)} = -\rho\,\overline{v_x' v_y'} = \mu^e \left[\frac{\partial \bar{v}_x}{\partial y} + \frac{\partial \bar{v}_y}{\partial x} \right]$$

in which the superscript "e" indicates "eddy" and μ^e is NOT a fluid property but rather it represents the action of turbulence on fluid motion.

<u>Observation</u> at the wall $\overline{v_i' v_j'} = 0$

and therefore $\mu^e = 0$

This implies that $\mu^e = \mu^e(y)$ where y is the wall distance in the reference case of channel/pipe flow.

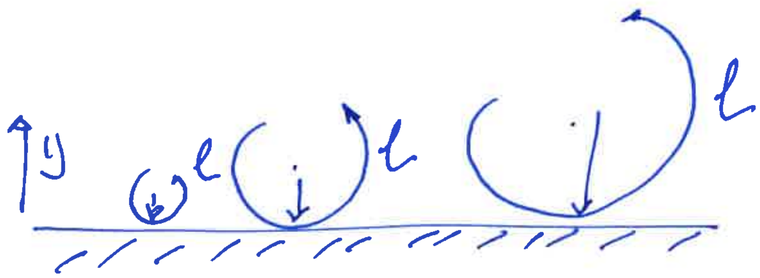
Prandtl proposed the following equation/model for the eddy viscosity:

$$\mu^e = \rho\, l^2 \left| \frac{d \bar{v}_x}{dy} \right|$$

where l = mixing length with the same physical meaning of the "mean free path" of molecules in the kinetic theory of gases.

In addition $l = l(y)$.

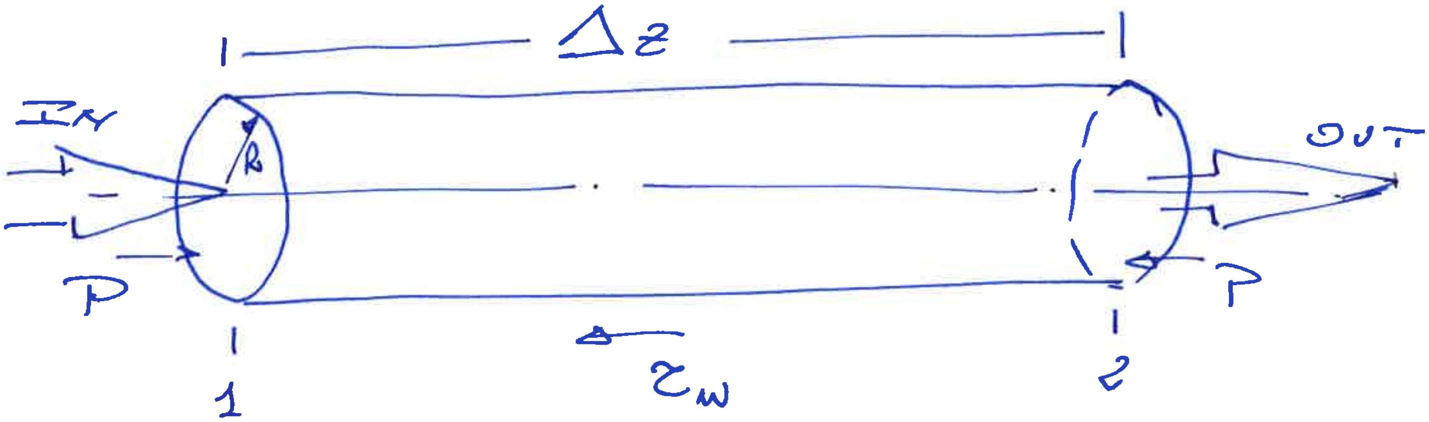
Prandtl hypothesized $l \propto y$ following the idea that the farther from the wall, the larger is the radius of the vortex which mixes the flow.



TURBULENT FLOW IN A PIPE

Calculation of Average Velocity

We consider the flow driven by a pressure gradient in a pipe (Poiseuille Flow) characterized by a Turbulent Reynolds number :



The force balance gives:

$$P_1 - P_2 = \Delta P = \frac{dP}{dz} \Delta z$$

$$\pi R^2 P_2 - \pi R^2 P_1 = \Delta z \cdot 2\pi R \, \tau_w$$

$$\cancel{\pi} R^{\cancel{2}} [P_2 - P_1] = \Delta z \cdot 2\cancel{\pi} \cancel{R} \, \tau_w$$

$$\boxed{\frac{\Delta P}{\Delta z} = -\frac{2}{R} \tau_w} \quad (A)$$

In cylindrical coordinates we have

$$\bar{v}_z = \bar{v}_z(r) \quad ; \quad \bar{v}_r = \bar{v}_\vartheta = 0 \quad ; \quad \frac{d\bar{P}}{dz} = \text{cost.}$$

The only relevant component of the Navier-Stokes Equation is the z_comp.

$$(NS_z) \quad \rho\left[\frac{\partial \bar{v}_z}{\partial t} + \bar{v}_r\frac{\partial \bar{v}_z}{\partial r} + \frac{\bar{v}_\vartheta}{r}\frac{\partial \bar{v}_z}{\partial \vartheta} + \bar{v}_z\frac{\partial \bar{v}_z}{\partial z}\right] =$$

$$= -\frac{d\bar{P}}{dz} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\bar{\tau}_{rz} + r\,\bar{\tau}^e_{rz}\right) + \frac{1}{r}\frac{\partial}{\partial \vartheta}\cancel{\left(\bar{\tau}_{\vartheta z} + \bar{\tau}^e_{\vartheta z}\right)} +$$

$$+ \frac{\partial}{\partial z}\cancel{\left(\bar{\tau}_{zz} + \bar{\tau}^e_{zz}\right)}$$

and finally

$$\boxed{0 = -\frac{d\bar{P}}{dz} + \frac{1}{r}\frac{d}{dr}r\left(\bar{\tau}_{rz} + \bar{\tau}^e_{rz}\right)}$$

~~Note~~ that it is exactly the same as in the Laminar case but with the presence of $\bar{\tau}^e_{rz}$.

Upon integration:

$$\boxed{\bar{\tau}_{rz} + \bar{\tau}^e_{rz} = \frac{r}{2}\frac{d\bar{P}}{dz}}$$

Eq. (A) is:

$$\frac{\Delta P}{\Delta z} = -\frac{2}{R}\tau_w \Rightarrow \boxed{\frac{d\overline{P}}{dz} = -\frac{2}{R}\overline{\tau}_w}$$

which substituted into the previous one:

$$\boxed{\overline{\tau}_{zz} + \overline{\tau}^e_{zz} = -\frac{2}{R}\overline{\tau}_w} \quad (B)$$

we know that: $\overline{\tau}_{zz} = \cancel{\mu}\frac{d\overline{v}_z}{dz}$

$$\overline{\tau}^e_{zz} = \rho\,\overline{v_z' v_z'}$$

and adopting the Prandtl mixing length model, we have:

$$\overline{\tau}^e_{zz} = \rho\,\overline{v_z' v_z'} = \underbrace{\rho\,l^2\left|\frac{d\overline{v}_z}{dz}\right|\frac{d\overline{v}_z}{dz}}_{(C)} = \cancel{\mu}^e\frac{d\overline{v}_z}{dz}$$

In pipe flow, turbulence is generated at the wall and we want to compute the velocity profile as a function of the wall distance, not of the eddies.

We apply a coordinate change as:

$$y = R - z \implies dy = -dz$$

$$z = R - y$$

Then, equation B plus C become:

$$\mu \frac{d\bar{v}_z}{dz} + \mu^e \frac{d\bar{v}_z}{dz} = -\frac{z}{R}\bar{\tau}_w$$

$$\mu \frac{d\bar{v}_z}{dz} + \rho \ell^2 \left|\frac{d\bar{v}_z}{dz}\right| \frac{d\bar{v}_z}{dz} + \frac{z}{R}\bar{\tau}_w = 0$$

and with the coordinate change:

$$-\mu \frac{d\bar{v}_z}{dy} - \rho \ell^2 \left|\frac{d\bar{v}_z}{dy}\right| \frac{d\bar{v}_z}{dy} + \frac{R-y}{R}\bar{\tau}_w = 0$$

Note: in the Prandtl's mixing length model for the viscosity the absolute value is necessary to avoid possible values of negative viscosity. However, in our case the derivative of the velocity profile (average) is always positive and we can safely remove the absolute value.

$$\Rightarrow \boxed{\rho \ell^2 \left(\frac{d\bar{v}_z}{dz} \right)^2 + \mu \frac{d\bar{v}_z}{dz} - \left(1 - \frac{y}{R}\right) \bar{\tau}_w = 0}$$

which is a first order differential equation non-linear in $\bar{v}_z$.

Boundary Conditions are:

$$\begin{cases} @\ y = 0 & \bar{v}_z = 0 \\ @\ y = R & \bar{v}_z = \bar{v}_{z,max} \end{cases}$$

As usual, we must now to make the equation dimensionless and we need to identify proper scaling variables.

We define:

<u>Shear velocity</u> $\quad u^* = \sqrt{\dfrac{\bar{\tau}_w}{\rho}}$

This velocity is characteristic of all processes occurring at the wall and is useful to define the <u>WALL VARIABLES</u>

velocity : $u^* = \sqrt{\dfrac{\bar{\tau}_w}{\rho}} \quad \Rightarrow \quad u^+ = \dfrac{\bar{v}_z}{u^*}$

length : $\ell^* = \dfrac{\nu}{u^*} \quad \Rightarrow \quad y^+ = \dfrac{y}{\nu / u^*}$

But we still have to define ℓ^+.
According with Prandtl's mixing length model:

Prandtl $\quad \ell^+ = k y^+ \quad$ with $k = 0{,}4$

Van Driest $\quad \ell^+ = k y^+ \left[1 - e^{-y^+/A}\right] \quad k = 0{,}4$

$A = 36$

→ This relation is empirically based and it recovers the Prandtl's equation for $y^+ >> A$

In equation (D) there is no reminiscence of the scale of the system except for R^+ and if $y^+ << R^+$ the term y^+/R^+ becomes negligible compared with 1. This has a physical meaning in the sense that turbulence is produced and controlled by what happens near the wall.
Mathematically, it can simplify the equation.

and our equation becomes:

$$\rho u^{*2} \ell^{+2} \left(\frac{du^+}{dy^+}\right)^2 + \mu \frac{u^*}{\ell^*} \frac{du^+}{dy^+} - \left(1 - y^+/R^+\right) \overline{\tau}_w = 0$$

$$\rho u^{*2} \ell^{+2} \left(\frac{du^+}{dy^+}\right)^2 + \rho \underbrace{\left(\frac{\nu}{\ell^*}\right)}_{u^*} u^* \frac{du^+}{dy^+} - \left(1 - y^+/R^+\right) \overline{\tau}_w = 0$$

remembering that $\overline{\tau}_w = \rho u^{*2}$ we have

$$\boxed{\ell^{+2}\left(\frac{du^+}{dy^+}\right)^2 + \frac{du^+}{dy^+} - \left(1 - y^+/R^+\right) = 0}$$

This equation is a quadratic form in $\frac{du^+}{dy^+}$ and the solution is

$$\frac{du^+}{dy^+} = \frac{-1 + \sqrt{1 + 4\ell^{+2}(1 - y^+/R^+)}}{2\ell^+}$$

upon integration:

$$\boxed{u^+ = \int_0^{y^+} \frac{-1 + \sqrt{1 + 4\ell^{+2}(1 - y^+/R^+)}}{2\ell^+}\, dy^+} \quad \text{B}$$

If we neglect the term y^+/R^+ we obtain:

$$u^+ = \int_0^{y^+} \frac{-1+\sqrt{1+4\ell^{+2}}}{2\ell^{+2}}\, dy^+$$

This equation depends only on the wall distance and therefore it can be generalized for any type of ~~w~~ turbulent flow over a wall (in a pipe or a channel). The solution of this equation will produce a UNIVERSAL PROFILE which means that measurements obtained for different geometries and for different Reynolds' numbers must overlap if plotted in terms of y^+ and u^+.

We will look for the solution of this equation in the two different regions of Inner flow (near the wall) and Outer flow (far from the wall)

Let us examine again our complete equation:

$$\text{(E)} \quad \underbrace{l^{+2}\left(\frac{du^+}{dy^+}\right)^2}_{\text{Reynolds Stress}} + \underbrace{\frac{du^+}{dy^+}}_{\text{Shear Stress}} - \underbrace{\left(1 - \frac{y^+}{R^+}\right)}_{\text{Pressure Gradient}} = 0$$

Reynolds Stress: Becomes important farther away from the wall

Shear Stress: dominates near the wall

Pressure Gradient

To neglect the first term (Reynolds's stress)

$l^+ << 1$ so that $l^{+2} <<< 1$

If we apply the Van Driest model we obtain:

$$y^+ = 10 \longrightarrow l^+ = 1 \longrightarrow l^{+2} = 1$$

$$y^+ = 5 \longrightarrow l^+ = 0{,}259 \longrightarrow l^{+2} = 0{,}067$$

makes negligible the R. stress Term.

So when we are very close to the wall the flow is viscosity dominated (Shear stress) and we call it <u>Viscous Sublayer</u>|

Solution for the Viscous Sublayer (Inner Region)

$y^+ \ll R^+$ and equation (E) becomes:

$$\frac{du^+}{dy^+} = 1 \rightarrow u^+ = \int_0^{y^+} dy^+$$

$\boxed{u^+ = y^+}$ which is reminiscent of Couette Flow

If we start from equation (D) and we apply $y^+ \ll R^+$

$$u^+ = \int_0^{y^+} \frac{-1 + \sqrt{1+4\ell^{+2}}}{2\ell^{+2}} dy^+ =$$

$$= \int_0^{y^+} \frac{2}{1+\sqrt{1+4\ell^{+2}}} dy^+ \simeq \int_0^{y^+} dy^+$$

$\ell^+ \ll 1$

Solution for the Inertial Layer

Turbulent Core (Outer Region)

Now we have to integrate from y^+ to R^+

$$u^+\Big|_{y^+}^{R^+} = \int_{y^+}^{R^+} \frac{-1+\sqrt{1+4\ell^{+2}(1-y^+/R^+)}}{2\ell^{+2}}\,dy^+$$

$$u^+ = u^+_{max} + \int_{y^+}^{R^+} \frac{1-\sqrt{1+4\ell^{+2}(1-y^+/R^+)}}{2\ell^{+2}}\,dy^+$$

For $\ell^+ \gg 1$

$$\Rightarrow 1-\sqrt{1+4\ell^{+2}(1-y^+/R^+)} \simeq -2\ell^+\sqrt{1-y^+/R^+}$$

and $$u^+ = u^+_{max} - \int_{y^+}^{R^+} \frac{\sqrt{1-y^+/R^+}}{\ell^+}\,dy^+$$

Of course, the same equation can be derived also from eq (E) when we neglect the shear stress term

$$\ell^{+2}\left(\frac{du^+}{dy^+}\right)^2 + \cancel{\frac{du^+}{dy^+}}^{\,0} - \left(1-\frac{y^+}{R^+}\right) = 0$$

In this outer region we can apply Prandtl's model : $\ell^+ = k y^+$

and we can solve the integral

$$u^+ = u^+_{max} - \int_{y^+}^{R^+} \frac{\sqrt{1 - y^+/R^+}}{\ell^+}\, dy^+ =$$

$$= u^+_{max} - \frac{1}{k} \underbrace{\int_{y^+}^{R^+} \frac{\sqrt{1 - y^+/R^+}}{y^+/R^+}\, d\,y^+/R^+}$$

if we set $x = y^+/R^+$ for convenience we must solve the integral:

$$\int \frac{\sqrt{1-x}}{x}\, dx$$

... solving this integral (a little tedious) we obtain :

$$\boxed{u^+ = u^+_{max} + \frac{2}{k}\sqrt{1 - y^+/R^+} + \frac{1}{k} \ln \frac{1 - \sqrt{1 - y^+/R^+}}{1 + \sqrt{1 - y^+/R^+}}}$$

We remark here that this solution is valid far from the wall (ln $\rightarrow$ diverges for $y^+ \rightarrow 0$)

Now, if y^+/R^+ is small as is when y^+ approaches R^+, then the square root can be approximated as:

$$\sqrt{1 - y^+/R^+} = 1 - \frac{1}{2}\, y^+/R^+ + \cdots$$

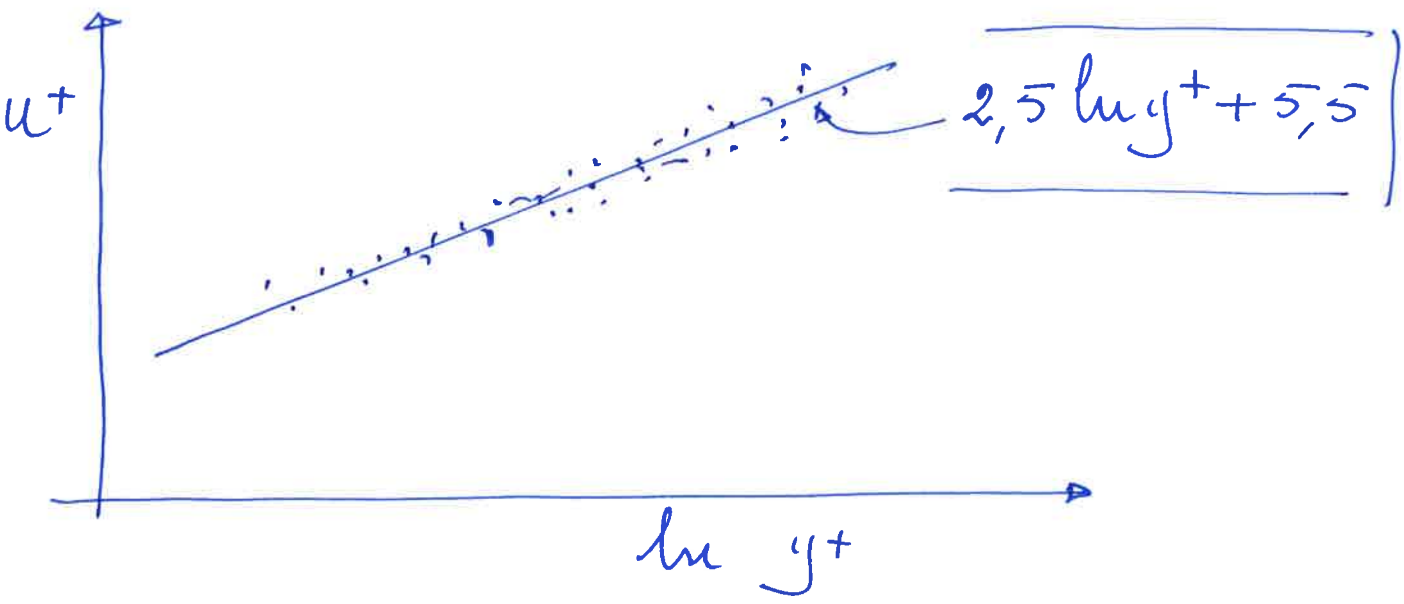
and then ($\ell^+ \gg 1$) we have

$$u^+ = \underbrace{u^+_{max}}_{\text{cost}} + \underbrace{\frac{2}{k}}_{\text{cost}} - \underbrace{\frac{1}{k}\, y^+/R^+}_{\text{Order } y^+/R^+} + \frac{1}{k}\ln y^+ + \cdots$$

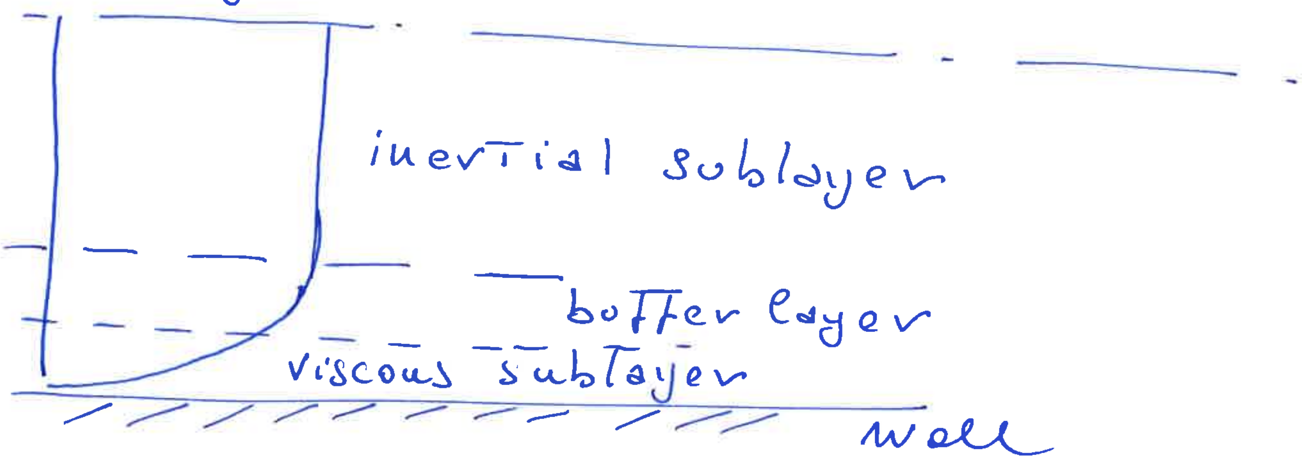
Finally:

$$\boxed{u^+ = \frac{1}{k}\ln y^+ + [\text{cost}] + O[y^+/R^+]}$$

and from experimental measurements



We have solved the equation in the following way



1) Viscous sublayer : only the viscous term is important

$$u^+ = y^+$$

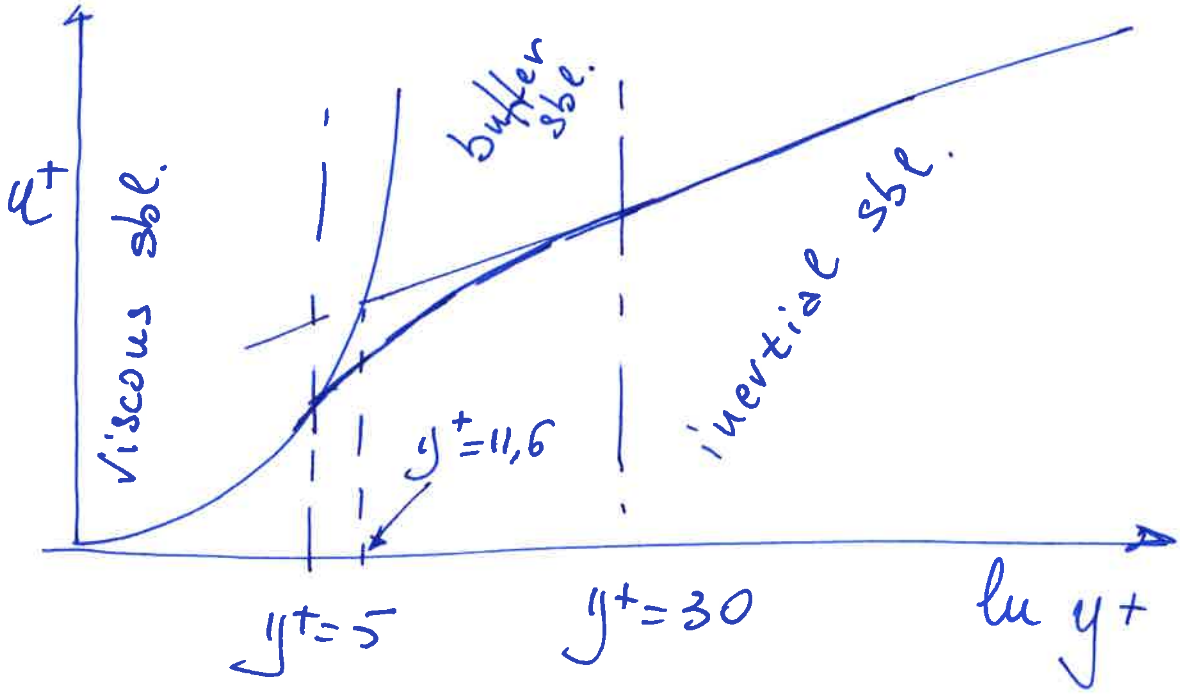
2) Buffer sublayer : both viscous and inertial term of the stresses are important

$$u^+ = ?$$

3) Inertial sublayer : only the inertial term of the stress is important

$$u^+ = 2{,}5 \ln y^+ + 5{,}5$$

So we must now try to plot the entire velocity profile.


u+
viscous sbl.
buffer sbl.
inertial sbl.
y+=11,6
y+=5
y+=30
ln y+