Modelling of Turbulent Flows

Lecture 3.2

March 24, 2020

Boundary Layer on a Flat Plate

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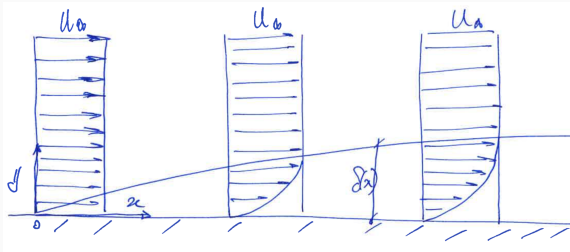
The available equations are

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

$$\rho \left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right) = \rho u_\infty \frac{\mathrm{d}u_\infty}{\mathrm{d}x} + \mu \frac{\partial^2 v_x}{\partial y^2}$$

Since u_∞ is constant,

$$\frac{\mathrm{d}u_\infty}{\mathrm{d}x} = 0$$

The problem in this case is rather complicated: we have two unknown dependent variables, v_x and v_y, which both depend on the two independent variables, x and y.

$$v_x(x, y) \neq 0 \quad ; \quad v_y(x, y) \neq 0$$

We can apply the similarity theory to reduce the dependence on two variables to the dependence on one function of two variables.

We find the same value of the velocity $\hat{v}$ at two different stations x_1 and x_2 for two different values of y, so we can identify a **similarity** function $\eta(x, y)$ so that
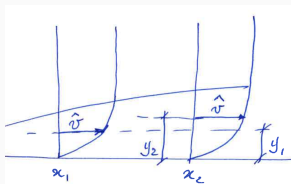
$$\hat{v}_x(\hat{\eta}) = \hat{v}[\hat{\eta}(x_1, y_1)] = \hat{v}[\hat{\eta}(x_2, x_2)]$$

However, in this case this is not enough, since we still have v_x and v_y. If we use the streamfunction we can however reduce the unknown variables to one; this at the price of *increasing the order of the differential equation*.

$$\begin{cases} \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \\ v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2} \end{cases}$$

BCs: $v_x = v_y = 0$ at $y = 0$; $v_x = u_\infty$ as $y \to \infty$.

NOTE: The similarity function is actually a similarity variable obtainsed just by rescaling the y coordinate with the thickness of the Boundary Layer. The concept is the **stretching** of the coordinate ($\eta \propto y/\delta(x)$).

Streamfunction

$$v_x = \frac{\partial \psi}{\partial y} \;\; ; \;\; v_y = \frac{\partial \psi}{\partial x}$$

Continuity (automatically satisfied):

$$-\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial x \partial y} = 0$$

NS_x:

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = -\nu \frac{\partial^3 \psi}{\partial y^3}$$

Defining the similarity variable as

$$\eta = \frac{y}{\delta(x)} = \frac{y}{\sqrt{\frac{\nu x}{u_\infty}}} = y\sqrt{\frac{u_\infty}{\nu x}}$$

Computing v_x from the definition of ψ,

$$\psi = f(\eta) \Rightarrow v_x = -\frac{\partial \psi}{\partial y} = -\frac{\partial f(\eta)}{\partial y} = -\frac{\mathrm{d}f}{\mathrm{d}\eta}\frac{\partial \eta}{\partial y} = -f'\sqrt{\frac{u_\infty}{\nu x}}$$

We have to avoid an explicit dependence of v_x on x. In addition, we would like to "fix" the dimensions of the streamfunction and we would rather have a dimensionless $f(\eta)$. We then define

$$\psi(\eta) := -\sqrt{\nu\, u_\infty\, x}f(\eta)$$

where f is dimensionless and $[\psi] = m^2/s$. We thus find that

$$v_x = -\frac{\partial \psi}{\partial y} = u_\infty\, f'\,\textcircled{1}$$

We need the other derivatives of ψ:

$$
\begin{aligned}
\frac{\partial \psi}{\partial x} &= v_y \\
&= -\frac{1}{2}\sqrt{\frac{\nu\, u_\infty}{x}}f - \sqrt{\nu\, u_\infty\, x}\frac{\mathrm{d}f}{\mathrm{d}\eta}\frac{\partial \eta}{\partial x} \\
&= -\frac{1}{2}\sqrt{\frac{\nu\, u_\infty}{x}}f - \sqrt{\nu\, u_\infty\, x}f' \cdot \left(-\frac{1}{2}\right)\frac{1}{x}\underbrace{y\sqrt{\frac{u_\infty}{\nu\, x}}}_{\eta} \quad ② \\
&= -\frac{1}{2}\sqrt{\frac{\nu\, u_\infty}{x}}f + \frac{1}{2}\sqrt{\frac{\nu\, u_\infty}{x}}\eta f'
\end{aligned}
$$

$$\frac{\partial^2 \psi}{\partial x \partial y} =$$

$$= \frac{\partial}{\partial x} \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial x} \left[-u_\infty f' \right]$$

$$= -u_\infty \frac{\mathrm{d} f'}{\mathrm{d} \eta} \frac{\partial \eta}{\partial x} \qquad \text{③}$$

$$= -u_\infty f'' \left(-\frac{1}{2} \right) \frac{y}{x} \sqrt{\frac{u_\infty}{\nu x}}$$

$$= \frac{1}{2} \frac{u_\infty}{x} \eta f''$$

$$\frac{\partial^2 \psi}{\partial y^2} = -u_\infty \, f'' \, \frac{\partial \eta}{\partial y} = -u_\infty \sqrt{\frac{u_\infty}{\nu \, x}} f'' \quad \text{④}$$

$$\frac{\partial^3 \psi}{\partial y^3} = -u_\infty \sqrt{\frac{u_\infty}{\nu \, x}} \, f''' \, \frac{\partial \eta}{\partial y} = -\frac{u_\infty^2}{\nu \, x} f''' \quad \text{⑤}$$

We need now to substitute ψ into the Navier-Stokes equation expressed in terms of ψ.

$$-u_\infty f' \frac{1}{2} \frac{u_\infty}{x} \eta f'' + \left[\frac{1}{2}\sqrt{\frac{\nu u_\infty}{x}} f - \frac{1}{2}\sqrt{\frac{\nu u_\infty}{x}} \eta f'\right] \cdot$$

$$\cdot \left(-u\infty \sqrt{\frac{u_\infty}{\nu x}} f''\right) = \frac{\nu u_\infty^2}{\nu x} f'''$$

$$\Rightarrow -\frac{u_\infty^2}{2x}\eta f' f'' + \frac{1}{2}\sqrt{\frac{\nu u_\infty}{x}}\left[f - \eta f'\right] \cdot \left(-u\infty\sqrt{\frac{u_\infty}{\nu x}} f''\right) = \frac{\nu u_\infty^2}{\nu x} f'''$$

$$\Rightarrow -\frac{1}{2}\frac{u_\infty^2}{x} f f'' = \frac{u_\infty^2}{x} f'''$$
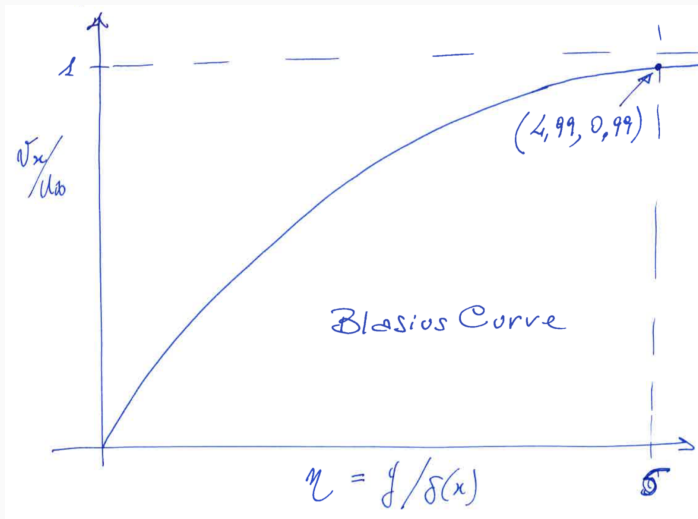
$$\Rightarrow \boxed{f''' + \frac{1}{2} f f'' = 0}$$

The equation to be solved is

$$\begin{cases} \eta = y\sqrt{\frac{u_\infty}{\nu x}} = \frac{y}{\delta(x)} \\ f''' + \frac{1}{2}f\,f'' = 0 \end{cases}$$

Boundary Conditions:

$$\begin{cases} f' = 0 & \text{at } \eta = 0 \\ f' = 1 & \text{for } \eta \to \infty \\ f = 0 & \text{at } \eta = 0 \end{cases}$$

Unfortunately this equation, although coming from an elegant
derivation, must be found by numerical integration.

Wall shear stress

We started all of this to compute the forces acting at the wall.
The wall shear stress is

$$\tau_w(x) = \mu \left.\frac{\partial v_x}{\partial y}\right|_{y=0} = -\mu \left.\frac{\partial^2 \psi}{\partial y^2}\right| y = 0 =$$

$$= \mu \, u_\infty \sqrt{\frac{u_\infty}{\nu x}} \left. f'' \right|_{y=0}$$

Numerically $\left. f'' \right|_{y=0} = f''(0) \simeq 0.332$, so

$$\tau_w(x) = 0.332 u_\infty \, \mu \sqrt{\frac{u_\infty}{\nu x}}$$

Boundary Layer Thickness

The thickness of the BL:

$$\delta^\star = \int\limits_0^\infty \left[1 - \frac{v_x(y)}{u_\infty}\right]\,\mathrm{d}y = 1.72\sqrt{\frac{\nu\,x}{u_\infty}}$$

The BL thickness increases with $\sqrt{x}$, while the wall shear stress decreases with $\frac{1}{\sqrt{x}}$: this is expected, given the decreasing slope of the velocity profile with x.

If we consider the case of a cylinder as an example:

in $\textcircled{A}$:

$$\frac{\partial P}{\partial x} < 0 \ ; \ \ \frac{\partial u_x}{\partial y} > 0 \ ; \ \ \tau_w = \mu \frac{\partial v_x}{\partial y} > 0 \ ; \ \ \omega_z = -\frac{\partial u_x}{\partial y} < 0 \ *$$

in $\textcircled{S}$:

$$\frac{\partial P}{\partial x} = 0 \ ; \ \ \frac{\partial u_x}{\partial y} = 0 \ ; \ \ \tau_w = \mu \frac{\partial u_x}{\partial y} = 0 \ ; \ \ \omega_z = 0$$
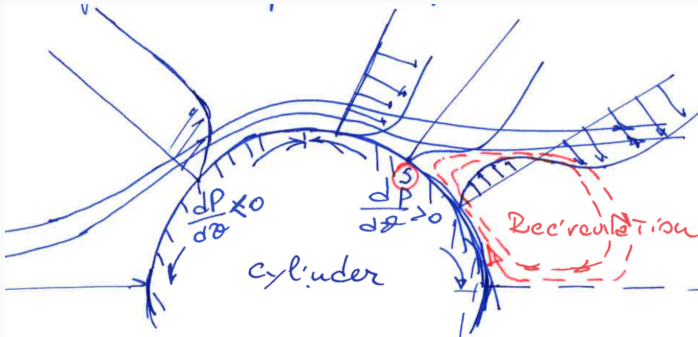
in $\textcircled{B}$:

$$\frac{\partial P}{\partial x} > 0 \ ; \ \ \frac{\partial u_x}{\partial y} < 0 \ ; \ \ \tau_w < 0 \ ; \ \ \omega_z > 0$$

$*$: clockwise rotation

It was not possible to predict the drag on a cylinder by means of the Potential Flow theory because this theory does not account for the fact that the Boundary Layer detaches from the sphere surface.

We can "straighten" the sphere/cylinder surface on a plane:
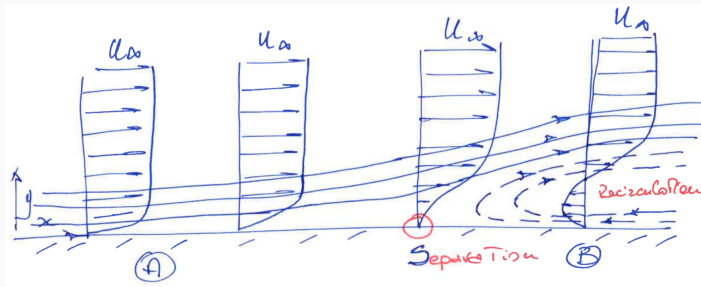
Why does the BL detach?

In case we have a flat plate, may the BL separate? **Maybe**.
The boundary layer thickness increases with x and the velocity
gradient decreases. If the velocity gradient reaches zero the
velocity profile hase a vertical derivative at the wall.

However, to change the direction of the velocity we need that the
second derivative of the velocity profile changes sign:

$$\left. \frac{\partial^2 v_x}{\partial y^2} \right|_{y=0} \quad (*)$$

Now, *at the wall* inertia is null and the NS_x becomes

$$-\frac{\partial P}{\partial x} + \mu \left(\underbrace{\cancel{\frac{\partial^2 v_x}{\partial x^2}}}_{*} + \frac{\partial^2 v_x}{\partial y^2} \right)$$

Then
$$\frac{\partial p}{\partial x} = \mu \left.\frac{\partial^2 v_x}{\partial y^2}\right|_{y=0}$$
and we remember that *derivpy* $= 0$, so the pressure behavior at the wall depends on what happens in the external flow. So **if** the pressure gradient in the external flow does not change sign, the *concavity* of the velocity profile may not happen and the boundary layer does not separate.

Classification of Boundary Layers

So what happens in the BL depends on what happens in the outer flow, and in particular on the behavior of the equation

$$\frac{\mathrm{d}p}{\mathrm{d}x} = -\rho \, u_\infty \, \frac{\mathrm{d}u_\infty}{\mathrm{d}x}$$

Classification of Boundary Layers

$$\frac{\mathrm{d}u_\infty}{\mathrm{d}x} > 0 \; ; \;\; \frac{\mathrm{d}p}{\mathrm{d}x} < 0$$

The pressure decreases with x and we can describe it as BL in a **favourable** pressure gradient. In this case, the BL is thin and the advection dominates the viscous diffusion, which is the cause of expansion of the BL. This BL has similar dynamics to the zero-pressure-gradient BL which we examined in great detail.

Classification of Boundary Layers

$$\frac{\mathrm{d}u_\infty}{\mathrm{d}x} < 0 \; ; \;\; \frac{\mathrm{d}p}{\mathrm{d}x} > 0$$

The pressure increases with x, and we speak of an **adverse** pressure gradient. In this case, the advection cannot work much and the thickness of the BL increases. This boundary layer is *prone to separation*.