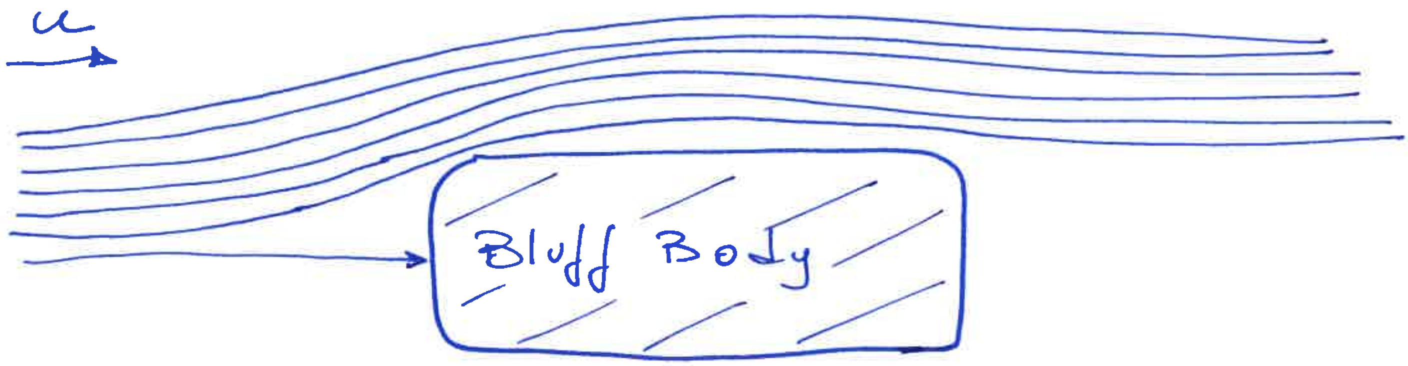
Vorlesung 8

Boundary Layers



U is the free stream velocity.

When a fluid in uniform flow @ velocity U meets with a bluff body, the speed at which disturbances (i.e. modifications to the flow) are transported downstream is U. However, the velocity at which disturbances are transported away from the wall (i.e. the velocity with which the information of the wall is transported to the outer flow) is proportional to the viscous diffusion velocity.

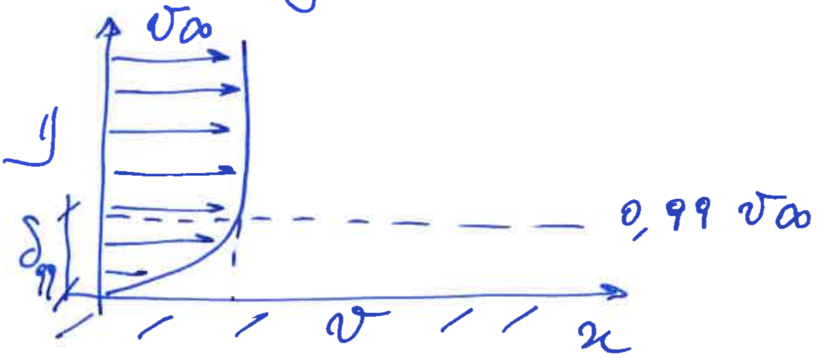
The perception is thus that disturbances are generated at the wall but are also rapidly transported downstream, leaving the FAR FIELD essentially undisturbed.

This is the ~~principle~~ concept of the Boundary Layer, which allows to confine the interactions body/fluid to a very thin region around the body.

■ PRACTICAL ENGINEERING DEFINITIONS

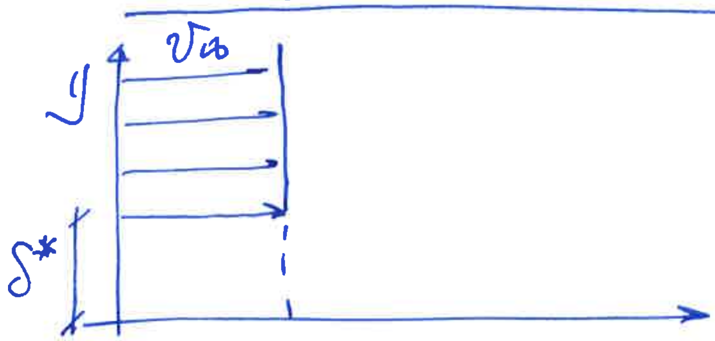
Thickness of the B.L.

(1)



δ_{99} is the distance from the wall at which the velocity reaches 99% of the free stream velocity.

(2) Displacement Thickness



δ^* is the thickness of a layer with Ø (zero) velocity which produces the same deficit

~~of mass~~

of mass transported downstream.

$$v_\infty \delta^* = \int_0^\infty v_\infty \, dy - \int_0^\infty v(y) \, dy =$$

$$= \int_0^\infty \left[v_\infty - v(y) \right] dy$$

$$\Rightarrow \delta^* = \int_0^\infty \left[1 - \frac{v(y)}{v_\infty} \right] dy$$

3 MOMENTUM THICKNESS

$\hat{\delta}$ is the thickness of a layer with zero velocity which produces the same defect of momentum transport downstream.

$$\rho \hat{\delta} v_\infty^{*2} = \rho \int_0^\infty v(y) \left[v_\infty - v(y) \right] dy$$

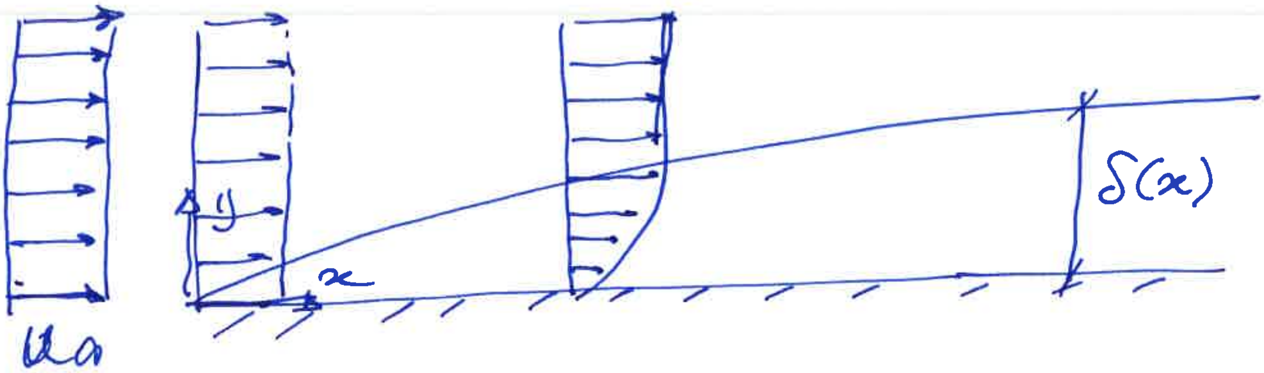
$$\Rightarrow \hat{\delta} = \int_0^\infty \frac{v(y)}{v_\infty} \left[1 - \frac{v(y)}{v_\infty} \right] dy$$

It is $\boxed{\delta_{99} > \delta^* > \hat{\delta}}$

BOUNDARY LAYER EQUATIONS

We want to derive the equations which are necessary to solve the velocity profile inside the boundary layer which forms when a free stream moving at velocity u_∞ meets a flat stationary plate.

We will hypothesize steady state flow over 2D ~~condit~~ geometry as in the sketch:



Continuity
$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

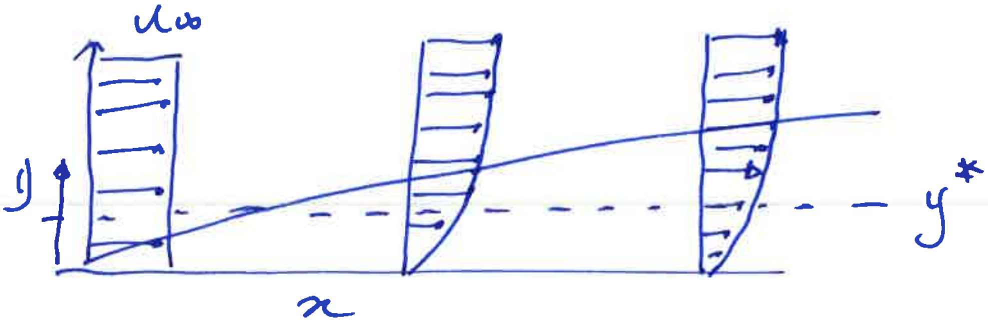
N-S)$_x$
$$\rho\left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right)$$

N-S)$_y$
$$\rho\left(v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y}\right) = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}\right)$$

We wish to make the equations dimensionless so to appreciate the order of magnitude of each term.

In this case we have no characteristic length. We choose $\delta(x)$ for the y direction and L (the length of the plate) for the x direction. u_∞ and V will be the characteristic velocities. δ and V are unknown.

- Continuity: $\dfrac{u_\infty}{L}\dfrac{\partial \tilde{v}_x}{\partial \tilde{x}} + \dfrac{V}{\delta}\dfrac{\partial \tilde{v}_y}{\partial \tilde{y}} = 0$



If we move along the line @ y^* we observe that ~~the~~ v_x changes and is therefore it is $v_x(x,y)$ - then both terms in the continuity equation must be of same order of magnitude and

$$\frac{u_\infty \cdot \delta}{L \cdot V} = O[1] \Rightarrow V = \frac{\delta}{L} u_\infty$$

with $\delta/L \ll 1$ and consequently $V/u_\infty \ll 1$

- $NS)_x$
$$\rho \frac{\delta^2 u_\infty}{L\mu}\left[\tilde{v}_x \frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \tilde{v}_y \frac{\partial \tilde{v}_x}{\partial \tilde{y}}\right] =$$

$$= -\frac{\Pi \delta^2}{\mu u_\infty L}\frac{\partial \tilde{P}}{\partial \tilde{x}} + \left[\left(\frac{\delta}{L}\right)^2 \frac{\partial^2 \tilde{v}_x}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{v}_x}{\partial \tilde{y}^2}\right]$$

with Π the scaling pressure

We learned that by definition, in the Boundary Layer inertial and viscous Terms must have the same order of magnitude and then

$$\rho \frac{u_\infty \delta^2}{L \mu} \simeq O[1] \Rightarrow \delta = \sqrt{\frac{\mu L}{\rho u_\infty}}$$

We can define a Reynolds number for the B.L. as:

$$\boxed{Re_L = \rho \frac{L u_\infty}{\mu}}$$

and then $\boxed{\delta = L \cdot Re_L^{-1/2}}$

and we find that pressure scales with inertial forces:

$$\Pi = \mu \frac{u_\infty L}{\delta^2} = \rho u_\infty$$

The Diffusion Term $(\delta/L)^2 \cdot \frac{\partial^2 \tilde{v}_x}{\partial \tilde{x}^2}$ is negligible and we have:

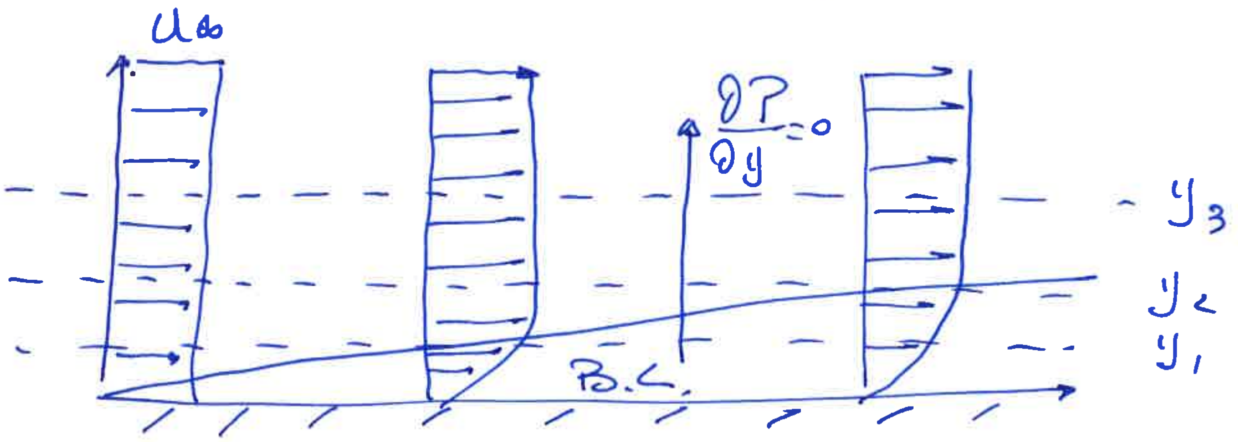
$$N/S)_x \qquad \boxed{\rho \left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right) = -\frac{\partial P}{\partial x} + \frac{\partial^2 v_x}{\partial y^2}}$$

The N/S)$_y$ component in dimensionless form is:

$$\left(\frac{\delta}{L}\right)^2\left[\tilde{v}_x\frac{\partial\tilde{v}_y}{\partial\tilde{x}}+\tilde{v}_y\frac{\partial\tilde{v}_y}{\partial\tilde{y}}\right]=-\frac{\partial\tilde{P}}{\partial\tilde{y}}+\left(\frac{\delta}{L}\right)^2\left[\left(\frac{\delta}{L}\right)^2\frac{\partial^2\tilde{v}_y}{\partial\tilde{x}^2}+\frac{\partial^2\tilde{v}_y}{\partial\tilde{y}^2}\right]$$

In which all terms except for the pressure gradient are negligible. Thus:

N/S)$_y$ $\quad \dfrac{\partial\tilde{P}}{\partial\tilde{y}}\cong 0$ $\quad$ which tells us that $\boxed{P=P(x)}$



Because the pressure does not depend on "y" but only on "x", we can determine the behavior of the pressure along y_3 with the potential flow equation for pressure (The Bernoulli equation)

Assuming no gravity effect:

$$p - \frac{1}{2}\rho u_\infty^2 = \text{const.}$$

$$\boxed{\frac{\partial P}{\partial x} = \frac{dP}{dx} = -\rho u_\infty \frac{du_\infty}{dx}}$$

The equations for the Boundary Layers are:

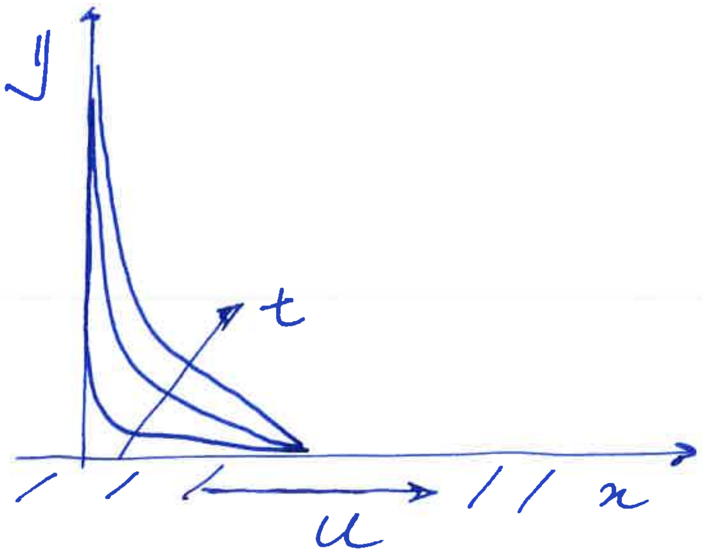
- Cont. $\dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y}$

- N/S $\rho\left[\dfrac{\partial v_x}{\partial t} + v_x \dfrac{\partial v_x}{\partial x} + v_y \dfrac{\partial v_x}{\partial y}\right] = \rho u_\infty \dfrac{du_\infty}{dx} + \mu \dfrac{\partial^2 v_x}{\partial y^2}$

with this term only when we want to study non-stationary B.L.

Stokes' Boundary Layer.*

We consider a flat plate, infinitely long, which is suddenly set in motion ~~at a~~



The plate is aligned with the "x" axis and the half infinite domain above is fluid which is still.

at $t=0$ the plate is set in motion ~~at a velo~~ in the "x" direction at velocity u.

The half domain is characterized by a velocity directed only along x.

$v_x \neq 0$ and $v_y = 0$

Of course, in this case v_x is not function of "x". Only of "y" and "t"

* We solve first this problem to become familiar with the similarity procedure

The continuity equation is

$$\frac{\partial v_x}{\partial x} + \cancel{\frac{\partial v_y}{\partial y}} = 0$$

The Navier-Stokes equation is

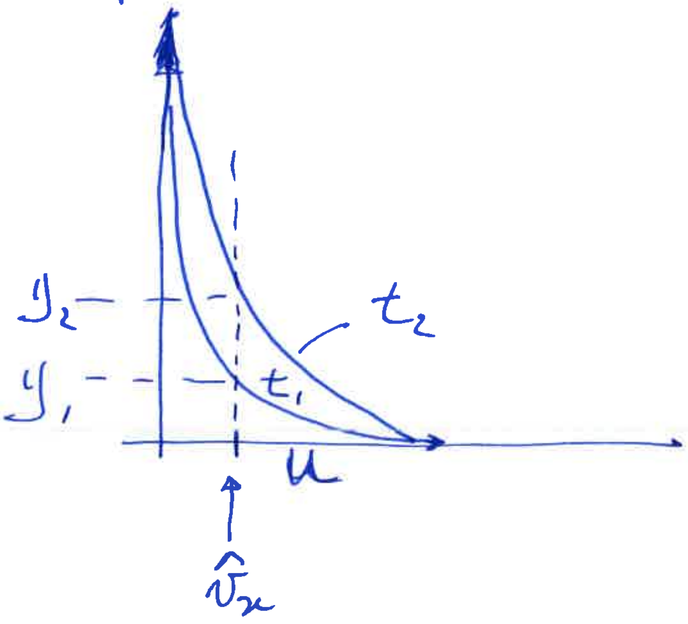
$$\rho \frac{\partial v_x}{\partial t} = \mu \frac{\partial^2 v_x}{\partial y^2}$$

and the Boundary Conditions are:

$$v_x(0, t) = \begin{cases} 0 & t \le 0 \\ U & t > 0 \end{cases} \qquad v_x(y \to \infty, t) = 0$$

In this problem there is no advection of momentum which is transferred along "y" just by the diffusion term. It is analogous to heat transfer from mass a wall or mass transfer, for instance (CO_2) absorption at a ~~[illegible]~~ film surface.

The solution is based on the similarity theory. The differential equation has one dependent variable which is a function of two independent variables (y and t).



We see that $\hat{v}_x(y_1, t_1) = \hat{v}_x(y_2, t_2)$

So, we look for a function η of y and t which has the following form: $\eta(y,t) = \alpha \, y / t^n$

with α and n to be determined.

For the similarity theory, it is:

$$\boxed{\frac{v_x(y,t)}{U} = f(\eta)}$$

On an intuitive basis this is justified as:

- If U is doubled, then we expect v_x to double, so v_x is ~~a linear function~~ proportional to U and its slope is given by $f(\eta)$

Our problem is now:

$$\begin{cases} v_x(y,t) = U \cdot f(\eta) & \text{(A)} \\[2ex] \underbrace{\dfrac{\partial v_x}{\partial t}}_{\mathrm{I}} = \nu \underbrace{\dfrac{\partial^2 v_x}{\partial y^2}}_{\mathrm{II}} & \text{(B)} \end{cases}$$

We substitute (A) in (B) –

The Term I becomes:

$$\frac{\partial v_x}{\partial t} = U \frac{df}{d\eta} \frac{\partial \eta}{\partial t} = f' \alpha (-n) \frac{y}{t^{n+1}} U =$$

$$\boxed{= -\frac{n}{t} \eta f' U}$$

The Term II:

$$\frac{\partial v_x}{\partial y} = U f' \frac{\partial \eta}{\partial y} = U f' \frac{\alpha}{t^n}$$

$$\frac{\partial^2 v_x}{\partial y^2} = U \frac{\alpha^2}{t^{2n}} f''$$

and substituting:

$$-\frac{n}{t}\eta f'\cancel{U} = \cancel{U}\frac{\alpha^2}{t^{2n}} f''\nu$$

We do not want an explicit dependence on "t" and therefore we choose $n = 1/2$

$$\boxed{\nu\alpha^2 f'' = -\eta/_2\, f'}$$

in which $\eta = \dfrac{\alpha y}{\sqrt{t}}$

Since η must be dimensionless, we choose

$$\alpha = \frac{1}{2\sqrt{\nu}} \Rightarrow \eta = \frac{1}{2}\frac{y}{\sqrt{\nu t}}$$

We put 2 because the solution becomes more convenient.

~~Thus~~

So the final equation is:

$$\boxed{f'' + 2\eta f' = 0}$$

which we solve as

$$\frac{df'}{d\eta} + 2\eta f' = 0 \Rightarrow \frac{df'}{f'} = -2\eta\, d\eta$$

$$d(\ln f') = -2\eta\, d\eta$$

$$\ln f' = -\eta^2 + \hat{C}_1$$

and
$$f'(\eta) = c_1 \exp[-\eta^2]$$
with
$$\boxed{\hat{c}_1 = \ln c_1}$$

and with a further integration:
$$f(\eta) = c_1 \int_0^{\eta} \exp[-\eta^2]\, d\eta + c_2$$

Boundary conditions are:
$$\begin{cases} \eta \to \infty & \Rightarrow f(\eta) = 0 \\ \eta \Rightarrow 0 & \Rightarrow f(0) = 1 \end{cases}$$
|| That is for $y \to \infty$ and for $t \to 0$

Applying the B.C. for $\eta = 0$
$$f(0) = c_1 \int_0^0 \exp[-\eta^2]\, d\eta + c_2 = 1$$

The integral behaves well and is zero and

Therefore $c_2 = 1$
$$f(\eta \to \infty) = c_1 \int_0^{\infty} \exp[-\eta^2]\, d\eta + 1$$

We know that $\int_0^{\infty} \exp[-\eta^2]\, d\eta = \frac{\sqrt{\pi}}{2}$

and then: $c_1 = -\frac{2}{\sqrt{\pi}}$

The velocity is thus:

$$\begin{cases} v_x(y,t) = U\left[1 - \frac{2}{\sqrt{\pi}}\int_0^{\eta} \exp[-\eta^2]\, d\eta\right] \\ \eta = \frac{y}{2\sqrt{\nu t}} \end{cases}$$

The function $\frac{2}{\sqrt{\pi}}\int_0^{\eta} \exp(-\eta^2)\, d\eta = \operatorname{erf}(\eta)$

Error function

$$\Rightarrow \boxed{v_x(y,t) = U\left[1 - \operatorname{erf}(\eta)\right]}$$