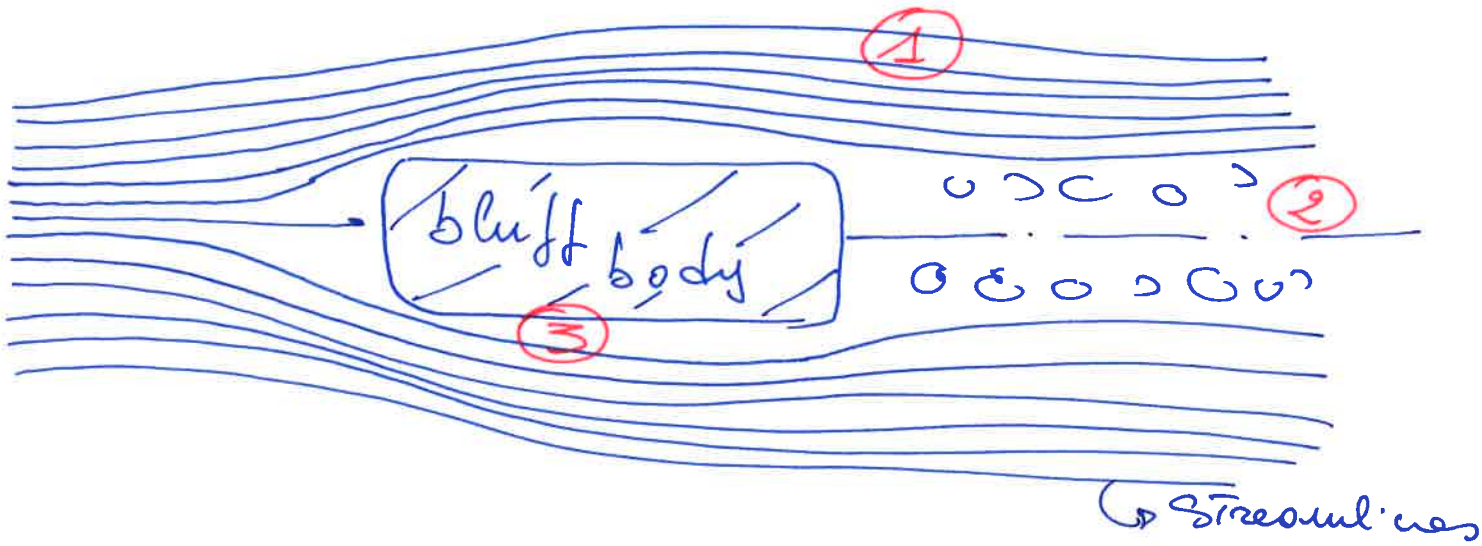
Vorlesung 6

Potential Flow

When we consider the flow of a fluid (with density ρ and viscosity μ) ~~...~~ past a bluff body we can analyze the flow field in the following way:



We can identify the three regions

(1) Potential Flow – Far from the body but with deformation of the streamlines

- Negligible viscous dissipation
- Vorticity $= 0$

2. Wake region (characterized by vortex shedding).

- Negligible viscous dissipation
- Vorticity non zero

3 Wall Region (Boundary Layer)

- Important viscous dissipation
- Vorticity Non zero

The velocity gradient near the wall is important due to the No-Slip Boundary condition. We observe high energy dissipation due to the big viscous stress

$$\tau_{ij} = \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

Definition of Vorticity

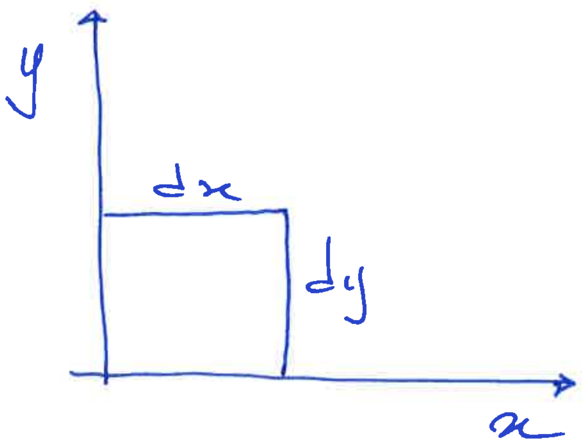
$$\vec{\omega} = \mathrm{rot}\,\vec{v} = \vec{\nabla} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} =$$

$$= \vec{i}\underbrace{\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)}_{\omega_x} + \vec{j}\underbrace{\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)}_{\omega_y} + \vec{k}\underbrace{\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)}_{\omega_z}$$

Vorticity represent the local rotation rate of an elementary parcel of fluid. Since vorticity is defined by the derivatives of the velocity vector, it is also related to the deformation rate.

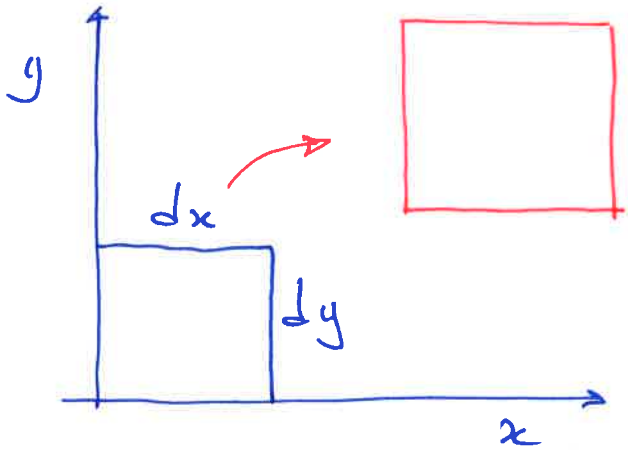
'NOT TO Lecture

- Definition of vorticity from the Rotation rate → kinematic derivation.

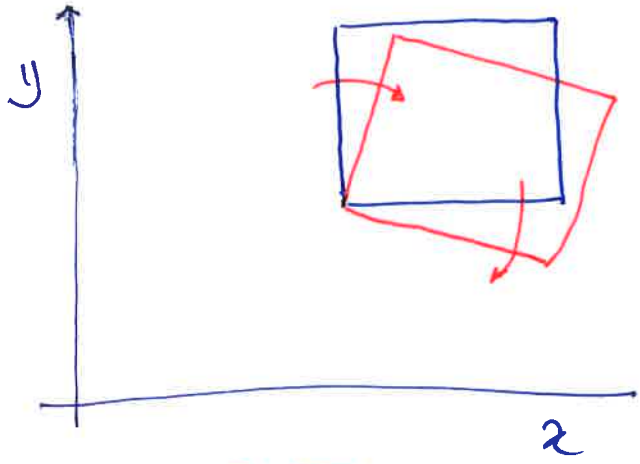


To simplify, we consider a 2D elemental fluid volume.

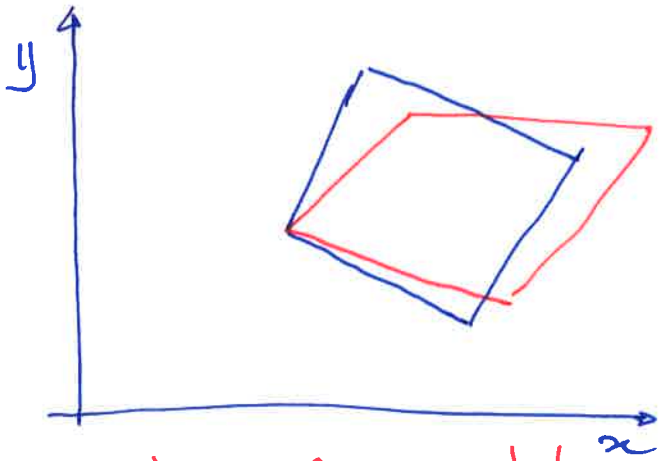
The elemental volume undergoes
Translation, rotation and angular deformation



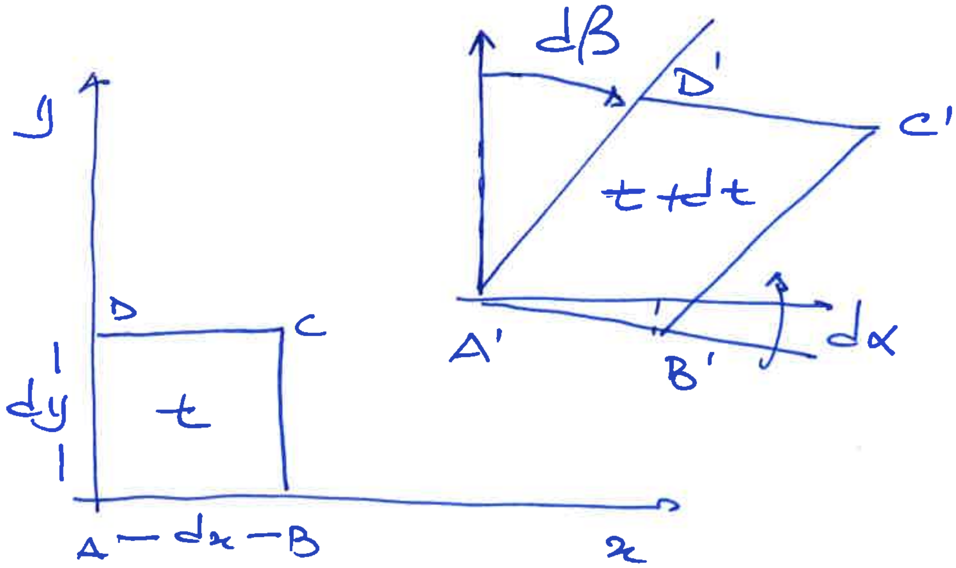
Translation



Rigid Rotation



Angular deformation



$d\beta \to$ clockwise ; $d\alpha \to$ counterclockwise

$$d\alpha = \tan^{-1}\left[\frac{(A'B')_y}{(A'B')_x}\right]$$

$$(A'B')_y = \left[v_y\Big|_{x+dx} dt - v_y\Big|_{x} dt \right]$$

$(A'B')_x \cong dx \quad \Leftarrow d\alpha <<$

in the limit of $d\alpha$ very small

Substituting : $d\alpha = \tan^{-1}\left[\frac{\partial v_y}{\partial x}\frac{dx}{dx}\, dt\right] \simeq$

$$\simeq \frac{\partial v_y}{\partial x} dt$$

$$\Rightarrow \quad \frac{d\alpha}{dt} = \dot{\alpha} = \frac{\partial v_y}{\partial x}$$

Analogous is the derivation of $\dot{\beta}$

$$\frac{d\beta}{dt} = \dot{\beta} = \frac{\partial v_x}{\partial y}$$

Therefore, the ~~xxxx~~ mean rotation rate (counterclockwise) is:

$$\frac{1}{2}\left[\dot{\alpha} - \dot{\beta}\right] = \frac{1}{2}\left[\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right] \quad *$$

Vorticity is related to the mean rotation rate. In 2D

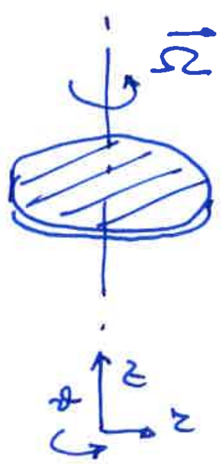
$$\omega = 2 \cdot \frac{1}{2}\left[\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right] = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}$$

Not to lecture

We can now try to make some examples to understand better the role of vorticity:

[1] Example: a fluid is rotating as if it were a rigid body with angular rotation rate Ω.

The velocity field is $\vec{u} = \vec{\Omega} \times \vec{r}$



$$\vec{\Omega} = \Omega_z \hat{k}$$

$$\vec{r} = x\hat{i} + y\hat{j}$$

$$\vec{v} = \Omega_z \hat{k} \times (x\hat{i} + y\hat{j}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \Omega_z \\ x & y & 0 \end{vmatrix} =$$

$$= \underbrace{\Omega_z \cdot x}_{v_y}\,\hat{j} - \underbrace{\Omega_z\, y}_{v_x}\,\hat{i} \qquad \text{and} \quad v_z = 0$$

The components of vorticity are:

$$\omega_x = \cancel{\frac{\partial v_z}{\partial y}}^{\,0} - \frac{\partial v_y}{\partial z} = -\frac{\partial}{\partial z}\left(\Omega_z \cdot x\right) = 0 \qquad (\Omega_z \cdot x \rightarrow \text{const.})$$

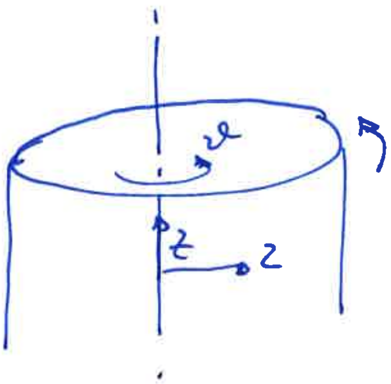
$$\omega_y = \frac{\partial v_x}{\partial z} - \cancel{\frac{\partial v_z}{\partial x}}^{\,0} = \frac{\partial}{\partial z}\left(\Omega_z \cdot y\right) = 0 \qquad (\Omega_z \cdot y \rightarrow \text{const})$$

$$\omega_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = \frac{\partial}{\partial x}\left(-\Omega_z y\right) - \frac{\partial}{\partial y}\left(\Omega_z x\right) =$$

$$= 2\,\Omega_z \neq 0$$

$\vec{\omega} = \omega_z \hat{k}$ [orthogonal to the motion plane]

$\boxed{2}$ Example Every fluid particle is moving on a circular path about the z-axis but with the radial velocity distribution corresponding to the Torsional flow.



$$v_\vartheta = \frac{k}{r} \qquad \text{with } k = \text{constant}$$

it is the flow generated ~~by a~~ in a cylindrical container by the boundary which moves at constant speed.

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r\, v_\theta) - \cancel{\frac{1}{r}\frac{\partial v_r}{\partial \theta}}^{\,=0} =$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{k}{r}\right) = \emptyset$$

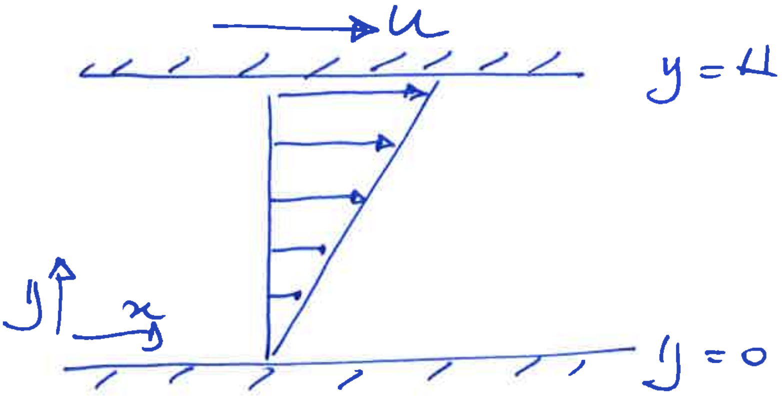
With ω_r and $\omega_\theta = 0$

$$\omega_r = \frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} = \emptyset$$

$$\omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} = 0$$

13 EXAMPLE

This is the simple shear flow in a Couette device.



$$v_x(y) = \frac{U}{H}\, y$$

$$\omega_x = \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} = 0$$

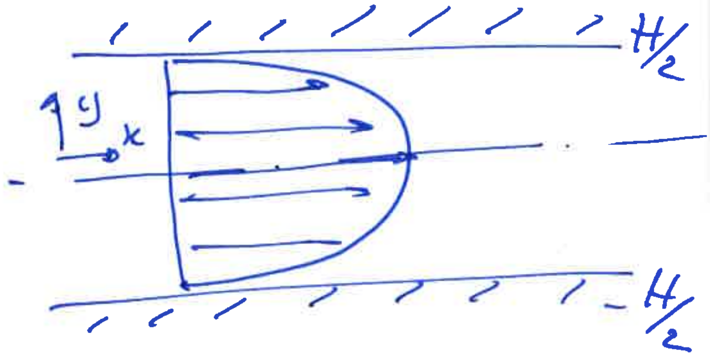
$$\omega_y = \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} = 0$$

$$\omega_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = -\frac{U}{H}$$

$\frac{U}{H}$ is, of course, the slope of the velocity profile.

$\boxed{4}$ Example – Plane Poiseuille flow

$$v_x(y) = \frac{1}{2\mu}\left(\frac{\Delta P}{L}\right)\left[y^2 - \frac{H^2}{4}\right]$$

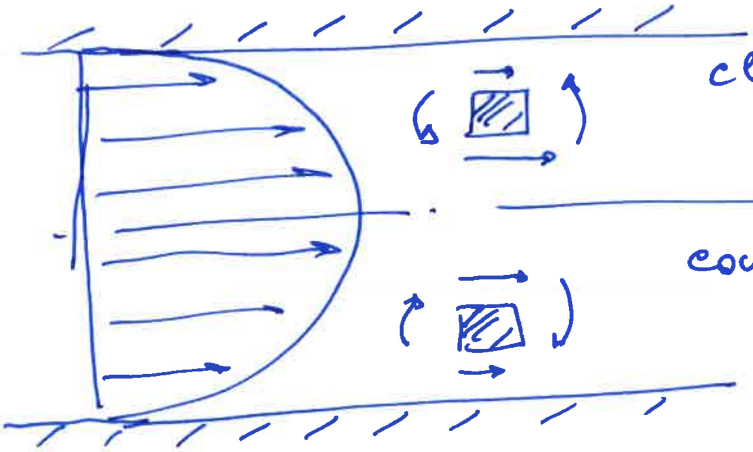


$$\omega_x = \omega_y = 0$$

$$\omega_z = -\frac{\partial v_x}{\partial y} =$$

$$= -\frac{1}{2\mu}\left(\frac{\Delta P}{L}\right) 2y = -\frac{1}{\mu}\frac{\Delta P}{L}\, y$$

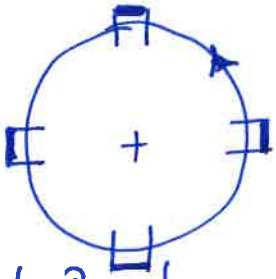
maximum vorticity (magnitude) is at both walls. Vorticity is zero in the centerplane.



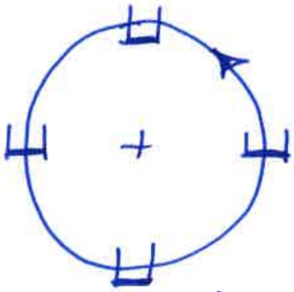
clockwise, negative vorticity

counterclockwise, positive vorticity

Discussion Examples $\boxed{1}$ & $\boxed{2}$



Rigid Body Rotation



Circulation w/o rotation

in the limit of ⊔ very small

VORTICITY TRANSPORT EQUATION

The vorticity transport equation describes the space and time evolution of vorticity. The equation is obtained by applying the curl operation to all terms of the Navier-Stokes equation.

① $$\mathrm{rot}\left[\rho\left(\frac{\partial\vec{v}}{\partial t}+\vec{v}\cdot\vec{\nabla}\vec{v}\right)=-\vec{\nabla}P+\mu\vec{\nabla}^2\vec{v}\right]$$

② $$\rho\left(\bar{\nabla}\times\frac{\partial\bar{v}}{\partial t}+\bar{\nabla}\times(\bar{v}\cdot\bar{\nabla}\bar{v})\right)=$$
$$=-\bar{\nabla}\times\bar{\nabla}P+\bar{\nabla}\times(\mu\nabla^2\bar{v})$$

③ $$\frac{\partial}{\partial t}(\bar{\nabla}\times\bar{v})+\bar{\nabla}\times(\bar{v}\cdot\bar{\nabla}v)=-\frac{\bar{\nabla}\times\bar{\nabla}P}{\rho}+\frac{\mu}{\rho}\bar{\nabla}\times\bar{\nabla}^2\bar{v}$$

This is the vorticity

We can simplify eq. ③ taking advantage of the following vector properties

I) ∀ scalar field A ⇒ $\bar{\nabla} \times \bar{\nabla} A = 0$

and → $\bar{\nabla} \times \bar{\nabla} P = 0$

II) $\bar{\nabla} \times \bar{\nabla} \bar{v} = \nabla^2 \bar{\omega}$

III) $\bar{\nabla} \times (\bar{v} \cdot \bar{\nabla} \bar{v}) = \bar{\nabla} \times \left[\bar{\nabla} \left(\underbrace{\frac{1}{2} \bar{v} \cdot \bar{v}}_{\frac{1}{2} v^2} \right) - \bar{v} \times \bar{\omega} \right] =$

$$= \underbrace{\cancel{\bar{\nabla} \times \left[\bar{\nabla} \left(\frac{1}{2} v^2 \right) \right]}}_{0} - \bar{\nabla} \times (\bar{v} \times \bar{\omega}) =$$

div $\bar{v} = 0$ [incomp.]

$\bar{\nabla} \cdot \bar{\omega} = \bar{\nabla} \cdot (\bar{\nabla} \times \bar{v}) = 0$ ∀ vector field

$$= - \bar{v} \, \cancel{(\bar{\nabla} \cdot \bar{\omega})} + \bar{\omega} \, \cancel{(\bar{\nabla} \cdot \bar{v})} +$$

0 if 2D

$$+ (\bar{v} \cdot \bar{\nabla}) \bar{\omega} - \cancel{(\bar{\omega} \cdot \bar{\nabla}) \bar{v}} =$$

$$= (\bar{v} \cdot \bar{\nabla}) \bar{\omega}$$

The vorticity transport equation becomes

$$\boxed{\frac{\partial \bar{\omega}}{\partial t} + (\bar{v} \cdot \bar{\nabla}) \bar{\omega} = \frac{1}{Re} \nabla^2 \bar{\omega}} \qquad 2D$$

$$\boxed{\frac{\partial \bar{\omega}}{\partial t} + (\bar{v} \cdot \bar{\nabla}) \bar{\omega} = (\bar{\omega} \cdot \bar{\nabla}) \bar{v} + \frac{1}{Re} \nabla^2 \bar{\omega}} \qquad 3D$$

and Re comes out by making properly dimensionless the equation.

This equation has three terms which indicate:

$$\boxed{\underbrace{\frac{\partial \bar{\omega}}{\partial t} + (\bar{v}\cdot\bar{\nabla})\bar{\omega}}_{\text{Material Derivative of } \bar{\omega}} = \underbrace{(\bar{\omega}\cdot\bar{\nabla})\bar{v}}_{\text{Vortex Stretching}} + \underbrace{\frac{1}{Re}\nabla^2\bar{\omega}}_{\text{Vorticity Diffusion}}}$$

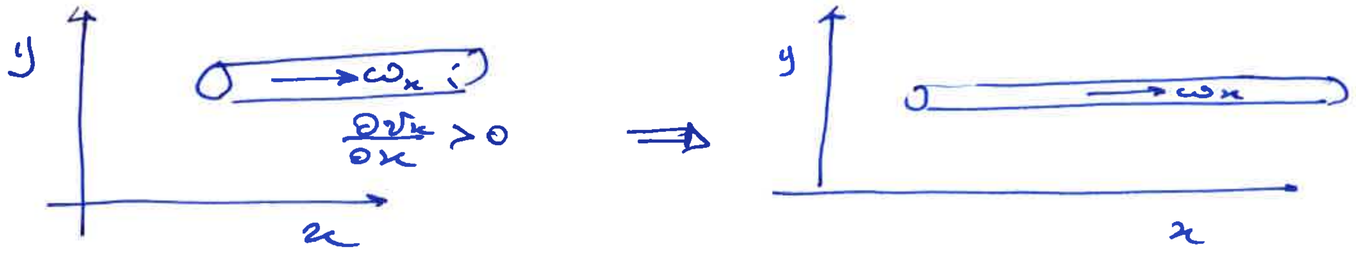
- Analysis of the Vortex stretching term

Considering for simplicity just one, x, component, the vortex stretching term is

$$x)\quad \left[(\bar{\omega}\cdot\bar{\nabla})\bar{v}\right]_x = \underbrace{\omega_x \frac{\partial v_x}{\partial x}} + \omega_y \frac{\partial v_x}{\partial y} + \omega_z \frac{\partial v_x}{\partial z}$$

→ This is the vortex stretching part of the term.

This part acts when a velocity gradient exists in the same direction of vorticity.

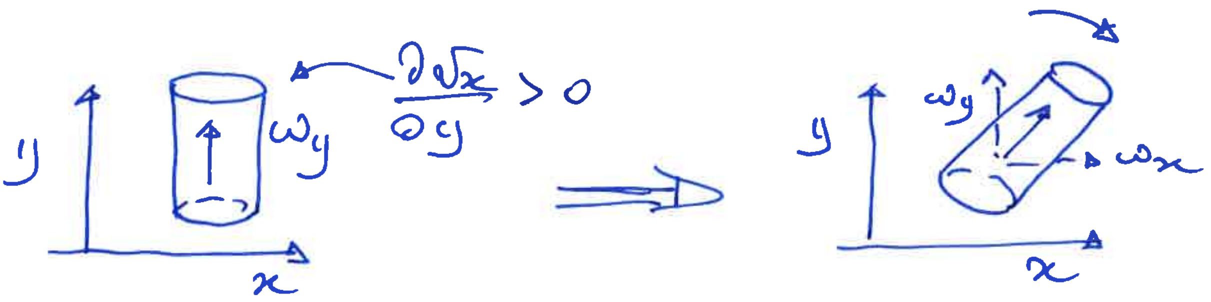


Due to this action, when the fluid parcel is stretched, then, to conserve the angular momentum, there will be a corresponding rotation rate increase and consequently an increase of vorticity. Much like the rotation speed of an ~~sk~~ ice-skate dancer.

<u>N.B.</u> This effect is very important in turbulence because it helps creating smaller scales.

This effect is an <u>autoamplification</u> effect. Just due to the alignment of velocity gradients and vorticity there is an increase in vorticity.

The other two terms, $\omega_y \frac{\partial v_x}{\partial y}$ and $\omega_z \frac{\partial v_x}{\partial z}$ contribute to rotate part of the existing vorticity and therefore to <u>transfer vorticity</u> from one component to the other



Note | Baroclinic Effect.

We have considered an incompressible fluid. If we allow difference in density we might have density gradients and the pressure term does not disappear from the Vorticity Equation which becomes:

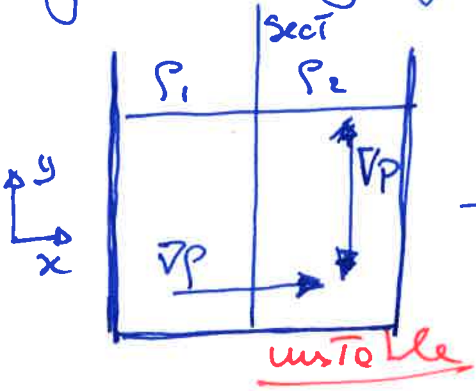
$$\frac{\partial \bar{\omega}}{\partial t} + (\bar{v}\cdot\bar{\nabla})\bar{\omega} = \boxed{\frac{\bar{\nabla}\rho \times \bar{\nabla}P}{\rho^2}} + \nu \nabla^2 \bar{\omega} + (\bar{\omega}\cdot\bar{\nabla})\bar{v}$$

Baroclinic Term

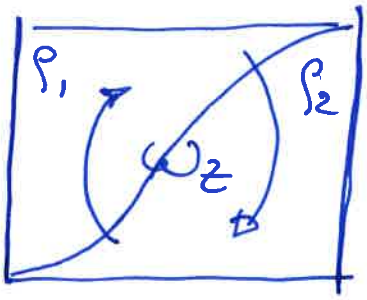
Usually, when density is allowed to vary, the density gradient is aligned with the pressure gradient (ocean density, atmospheric density in stable conditions). However, situations may arise when these two gradients are not aligned and vorticity is produced.

Consider the following example:

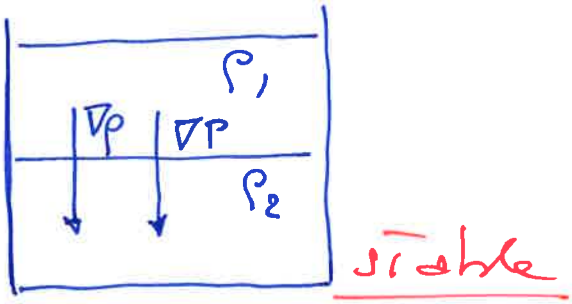
In a container we have low and high density fluid separated by a sect.



Sect is removed



@ steady state



Potential Flow and Two-Dimensional Vorticity equation

In a steady, 2D incompressible flow the vorticity equation is

$$\frac{\partial \overline{\omega}}{\partial t} = 0 \Rightarrow \boxed{(\overline{v} \cdot \overline{\nabla})\overline{\omega} = \frac{1}{Re} \nabla^2 \overline{\omega}}$$

Assuming viscous dissipation negligible, we have $\nu = 0 \rightarrow Re \rightarrow \infty$ and

$$\boxed{(\overline{v} \cdot \overline{\nabla}) \cdot \overline{\omega} = 0}$$

In 2D $\overline{\omega} \equiv \omega$

and $\overline{v} \cdot \overline{\nabla} \omega = 0$ which implies that $\overline{v} \perp \overline{\nabla} \omega$

Kelvin's Theorem states that in an inviscid fluid ($\nu = 0$) the circulation of a material tube is constant.

The circulation is $\boxed{\Gamma = \int \omega \, dS}$

where dS is the differential surface of a material tube.

If we are far from the body and the fluid is irrotational (i.e $\omega = 0$) in that region, Kelvin's theorem states that the flow field is irrotational everywhere.

And if the flow is irrotational in the entire domain, we can describe the flow field by a suitable function which is called potential

POTENTIAL FUNCTION, φ

~~The~~ which is a scalar function (can be defined in 2 or 3D being an Hamiltonian).

The potential function must satisfy the equation $\boxed{\bar{v} = -\bar{\nabla}\varphi}$

"$-$" is by convention

$$v_x = -\frac{\partial \varphi}{\partial x} \qquad v_y = -\frac{\partial \varphi}{\partial y}$$