11

BOUNDARY LAYERS, WAKES, AND JETS

11.1 Viscous regions in high Reynolds number flow

The reason for the occurrence of boundary layers and their role in high Reynolds number flows have been considered in Section 8.3. Now that we have considered (in Chapter 10) the flow external to the boundary layers, we need to look in more detail at the boundary layers themselves. This is the first purpose of this chapter. However, wakes and jets—other regions where viscous forces are significant even at high Reynolds number—involve similar ideas and equations, and so we extend the discussion to include them.

The fact that flow outside the boundary layers is irrotational (Section 10.3) provides another way of viewing the process of boundary layer formation. Fluid particles can acquire vorticity only by viscous diffusion (i.e. through the action of the term $\nu\nabla^2\boldsymbol{\omega}$ in eqn (6.27)). The action of viscosity comes in at the boundary through the need to satisfy the no-slip condition. As a result vorticity is introduced into the flow at the boundary, and then diffuses away from it. The boundary layer can be defined as the region of appreciable vorticity. The boundary layer is long and thin ($L \gg \delta$) when the fluid travels a long distance downstream during the time that the vorticity diffuses only a small distance away from the boundary. This happens when the Reynolds number is large.

High Reynolds number wakes and jets are regions into which vorticity has been advected; the vorticity was introduced upstream where the fluid was close to boundaries—the walls of the obstacle producing the wake or of the orifice through which the jet emerges. These flow features are long and thin for just the same reason as a boundary layer is.

11.2 The boundary layer approximation [23]

Because of the difference in length scales in different directions, certain terms in the equations of motion play a negligible part in the dynamics of boundary layers. We now see in a systematic way how this can be used to formulate an appropriate approximation to the equations. This will provide further justification for the ideas introduced in Section 8.3. Also, the resulting equations can sometimes be solved when the exact equations cannot. We shall be looking (Sections 11.4 and 11.6) at a couple of solutions both for their own interest and as our examples of the mathematical methods used for fully non-linear problems.

From the outset we confine attention to steady, two-dimensional boundary layers—a severe restriction from a practical point of view, but one that still allows us to see the general principles involved. We suppose that the boundary layer is forming on a flat wall (with the x-coordinate in the flow direction and y normal to the wall). A free-stream velocity outside the boundary layer is prescribed as a function of x. This could be achieved by making the wall one side of a channel of variable width as in Fig. 11.1 (with the channel width always large compared to the boundary layer thickness). In fact, however, it makes negligible difference if the surface is curved, so long as there are no sharp corners—more precisely, so long as the radius of curvature of the surface is everywhere large compared to the boundary layer thickness. Thus, the prescribed free-steam velocity could be a solution of Euler's equation for flow past an obstacle (x then being a curvilinear coordinate in the surface).

We denote the free-stream velocity by u_0 and the pressure associated with it by p_0.

We take the boundary layer to have length scales L and δ in the x- and y-directions, as in Section 8.3. We may expect that the velocity scales will also be different in different directions and we denote the scales of u and v by U and V. Similarly the order of magnitude of the pressure differences across the boundary layer in the y-direction may not be the same as the order of magnitude of the imposed pressure differences outside the boundary layer; we denote the scale of the former by Λ and the scale of the latter by Π. We now consider each of the equations in turn, labelling the terms with their orders of magnitude.

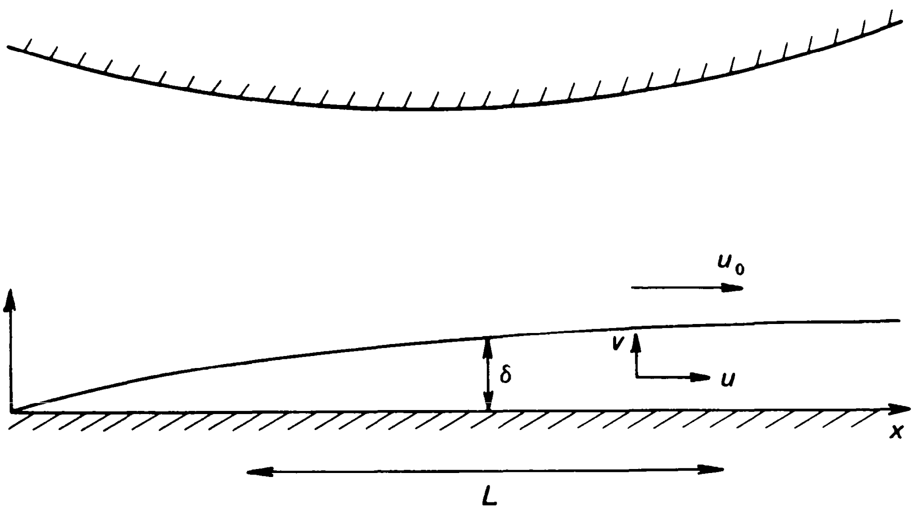

FIG. 11.1 Boundary layer on flat wall of channel: definition sketch.

The continuity equation is

$$\begin{array}{cc} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\ \dfrac{U}{L} \quad \dfrac{V}{\delta}. \end{array} \tag{11.1}$$

The two terms must be of the same order of magnitude; fluid entering or leaving the boundary layer at its outer edges must be associated with variations in the amount of fluid travelling downstream within the boundary layer. Hence,

$$V \sim U\delta/L; \tag{11.2}$$

the velocity component normal to the wall is small compared with the rate of downstream flow when the boundary layer is thin.

The x-component of the Navier–Stokes equation is

$$\begin{array}{cccccc} u\dfrac{\partial u}{\partial x} + & v\dfrac{\partial u}{\partial y} & = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + & \nu\dfrac{\partial^2 u}{\partial x^2} + & \nu\dfrac{\partial^2 u}{\partial y^2} \\ \dfrac{U^2}{L} & \dfrac{VU}{\delta} \sim \dfrac{U^2}{L} & \dfrac{\Pi}{\rho L} & \dfrac{\nu U}{L^2} & \dfrac{\nu U}{\delta^2}. \end{array} \tag{11.3}$$

The second expression for the order of magnitude of $v\,\partial u/\partial y$ has been written using relationship (11.2). The two parts of the inertia term are comparable with one another, the smallness of V/U compensating for the more rapid variation of u with y than with x. The two parts of the viscous term are however of different sizes when δ/L is small, and $\nu\,\partial^2 u/\partial x^2$ may be neglected.

The y-component of the Navier–Stokes equation is

$$\begin{array}{cccccc} u\dfrac{\partial v}{\partial x} & + \quad v\dfrac{\partial v}{\partial y} & = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} + & \nu\dfrac{\partial^2 v}{\partial x^2} & + \nu\dfrac{\partial^2 v}{\partial y^2} \\ \dfrac{UV}{L} \sim \dfrac{U^2\delta}{L^2} & \dfrac{V^2}{\delta} \sim \dfrac{U^2\delta}{L^2} & \dfrac{\Lambda}{\rho\delta} & \dfrac{\nu V}{L^2} \sim \dfrac{\nu U\delta}{L^3} & \dfrac{\nu V}{\delta^2} \sim \dfrac{\nu U}{L\delta}. \end{array} \tag{11.4}$$

In both eqn (11.3) and eqn (11.4) the pressure term will be of the same order of magnitude as the largest of the other terms. Hence,

$$\Pi/\rho L \sim U^2/L \sim \nu U/\delta^2 \tag{11.5}$$

$$\Lambda/\rho\delta \sim U^2\delta/L^2 \sim \nu U/L\delta \tag{11.6}$$

and so

$$\Lambda/\Pi \sim \delta^2/L^2. \tag{11.7}$$

The pressure differences across the boundary layer are much smaller than those in the x-direction. Hence, at any value of y the difference between $(1/\rho)\,\partial p/\partial x$ and $(1/\rho)\,dp_0/\mathrm{d}x$ is much smaller than the significant terms in eqn (11.3) and we may replace the former by the latter, giving

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\mathrm{d}p_0}{\mathrm{d}x}+\nu\frac{\partial^2 u}{\partial y^2}. \tag{11.8}$$

Outside the boundary layer there is no variation with y and

$$u_0\frac{\mathrm{d}u_0}{\mathrm{d}x}=-\frac{1}{\rho}\frac{\mathrm{d}p_0}{\mathrm{d}x}, \tag{11.9}$$

a result which could also be obtained from Bernoulli's equation. Hence

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=u_0\frac{\mathrm{d}u_0}{\mathrm{d}x}+\nu\frac{\partial^2 u}{\partial y^2}. \tag{11.10}$$

This equation together with

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{11.11}$$

constitute the boundary layer equations—two equations in the two variables u and v.

11.3 Classification of boundary layers

Solution of (11.10) and (11.11) requires $u_0(x)$ to be specified, both to give the third term in (11.10) and as a boundary condition for integration with respect to y. This is why the solution of Euler's equation for the particular configuration is needed before the boundary layer can be analysed.

Obviously, many different distributions of $u_0(x)$ can arise. In the next section we shall consider the simplest case of all—when u_0 is constant. We shall not consider any other case quantitatively, but some general remarks may be made. A useful broad classification is given by the sign of $\mathrm{d}u_0/\mathrm{d}x$ or, equivalently through eqn (11.9), the sign of $\mathrm{d}p_0/\mathrm{d}x$. When

$$\mathrm{d}u_0/\mathrm{d}x>0;\quad \mathrm{d}p_0/\mathrm{d}x<0 \tag{11.12}$$

(the external flow is accelerating as the pressure decreases) one talks of a boundary layer in a favourable pressure gradient. When

$$\mathrm{d}u_0/\mathrm{d}x<0;\quad \mathrm{d}p_0/\mathrm{d}x>0 \tag{11.13}$$

(the external flow is decelerating as the pressure rises) one talks of an adverse pressure gradient. One can, of course, have regions of each type of pressure gradient within a given flow—indeed, this is usually the case for the boundary layer on an obstacle.

Boundary layers in favourable pressure gradients are relatively thin. In a region of strong enough pressure gradient the boundary layer thickness can actually decrease with distance downstream; the effect of the pressure gradient more than counteracts the viscous spreading process (explained qualitatively in Section 11.1 and to be seen quantitatively in Section 11.4). We shall also be noting in Section 18.2 that instability, leading to transition to turbulence, is delayed by a favourable pressure gradient. Such a pressure gradient does not, however, introduce flow phenomena qualitatively different from those occurring in boundary layers with zero pressure gradient.

The effects of an adverse pressure gradient are in the first place just the reverse of those just described. Much more significantly, however, a boundary layer in such a pressure gradient is prone to the phenomenon of separation. We shall discuss the nature and implications of this in Chapter 12 and particularly Sections 12.4 and 12.5. It should be noted here, however, that the effect of separation can be to modify the solution of Euler's equation for the region outside the boundary layer. Consequently, $u_0(x)$ may differ from the form that one initially assumes.

11.4 Zero pressure gradient solution

The simplest, and in a sense most fundamental, case is the one where the pressure gradient is zero. Equivalently, u_0 is constant; we consider the boundary layer beneath a uniform flow. Such a boundary layer is readily observed on a thin flat plate set up parallel to the free-stream; one wall of an empty wind-tunnel or water-channel is sometimes used.

The equations for this case are

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2} \tag{11.14}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{11.15}$$

with boundary conditions

$$u=v=0 \quad \text{at} \quad y=0$$

$$u\to u_0 \quad \text{as} \quad y\to\infty. \tag{11.16}$$

We look for a solution of the form

$$u=u_0 g(y/\Delta) \tag{11.17}$$

where Δ is a function of x. That the solution should be of this form is an assumption. It corresponds to the velocity profile having the same shape at all values of x, although with a different scale in the y-direction, and is thus physically plausible. Δ is directly proportional to the boundary layer thickness, but it is convenient to define it slightly differently from δ.

Equation (11.15) can be satisfied by introducing a stream function ψ such that

$$u = \partial\psi/\partial y, \quad v = -\partial\psi/\partial x \tag{11.18}$$

as in Section 6.3. If we take

$$\psi = u_0 \Delta f(y/\Delta) \tag{11.19}$$

then (11.18) gives (11.17) as required with

$$g = f' \tag{11.20}$$

where the prime indicates differentiation with respect to

$$\eta = y/\Delta. \tag{11.21}$$

The second of equations (11.18) also gives

$$v = u_0(-f + yf'/\Delta)\,\mathrm{d}\Delta/\mathrm{d}x \tag{11.22}$$

and further differentiation leads to

$$\frac{\partial u}{\partial x} = -\frac{u_0 y f''}{\Delta}\frac{\mathrm{d}\Delta}{\mathrm{d}x}; \quad \frac{\partial u}{\partial y} = \frac{u_0 f''}{\Delta}; \quad \frac{\partial^2 u}{\partial y^2} = \frac{u_0 f'''}{\Delta^2}. \tag{11.23}$$

Substitution into eqn (11.14) then gives

$$\frac{u_0^2}{\Delta}\frac{\mathrm{d}\Delta}{\mathrm{d}x} ff'' + \frac{\nu u_0}{\Delta^2} f''' = 0. \tag{11.24}$$

If the solution is of the assumed form this must reduce to a total differential equation in f as a function of η; i.e. the two coefficients must have the same dependence on x, so that this cancels out:

$$\frac{u_0^2}{\Delta}\frac{\mathrm{d}\Delta}{\mathrm{d}x} \propto \frac{\nu u_0}{\Delta^2} \tag{11.25}$$

and so

$$\Delta^2 \propto \nu x/u_0 + \text{const.} \tag{11.26}$$

It is convenient to choose the constant of proportionality and the origin of x so that

$$\Delta = (\nu x/u_0)^{1/2}. \tag{11.27}$$

It is found experimentally that this choice of the origin of x corresponds fairly closely to the leading edge of a flat plate set up in an otherwise unobstructed flow. Equation (11.27) is essentially the same result as eqn (8.13).

Equation (11.24) now becomes

$$ff'' + 2f''' = 0. \tag{11.28}$$

The boundary conditions transform to

$$\begin{aligned} f = f' = 0 \quad &\text{at} \quad \eta = 0 \\ f' \to 1 \quad &\text{as} \quad \eta \to \infty. \end{aligned} \tag{11.29}$$

The solution of this total differential equation has to be obtained numerically [22, 23]. The resulting variation of f' with η, and so the velocity profile is shown in Fig. 11.2. This curve is known as the Blasius profile.

It has the property that

$$f' = 0.99 \quad \text{when} \quad \eta = 4.99. \tag{11.30}$$

The boundary layer thickness as previously defined (Section 8.3) is thus

$$\delta = 4.99(\nu x/u_0)^{1/2}. \tag{11.31}$$

Other ways of writing this are

$$\delta/x = 4.99\,\mathrm{Re}_x^{-1/2} \quad \text{and} \quad \mathrm{Re}_\delta = 4.99\,\mathrm{Re}_x^{1/2} \tag{11.32}$$

($\mathrm{Re}_x = u_0 x/\nu$; $\mathrm{Re}_\delta = u_0\,\delta/\nu$). The boundary layer thickness is small when the Reynolds number is large, as expected. This is, of course, a necessary condition for the theory to apply. Also Re_δ is large when Re_x is large; there is no ambiguity in talking about large Reynolds number.

Figure 11.2 includes experimental observations for several values of Re_x (from two separate experiments). The agreement with the theoretical profile is good, providing support for the various approximations and assumptions made in the course of the theory. The experimental results have been scaled to the coordinates $\eta(=y(u_0/\nu x)^{1/2})$ and $f'(=u/u_0)$. One sees the way in which the profile maintains its shape with distance downstream although the boundary layer thickness is changing—as assumed in eqn (11.17).

At higher values of the Reynolds number, the Blasius profile is unstable and the boundary layer becomes turbulent. The transition process will be described in Chapter 18, and the nature of the turbulent boundary layer in Chapter 21. The instability depends on Re_δ, which, as we see from eqn (11.32), increases with Re_x. Thus, any zero pressure gradient boundary layer undergoes transition if it extends far enough downstream. However, provided that the disturbance level is not too

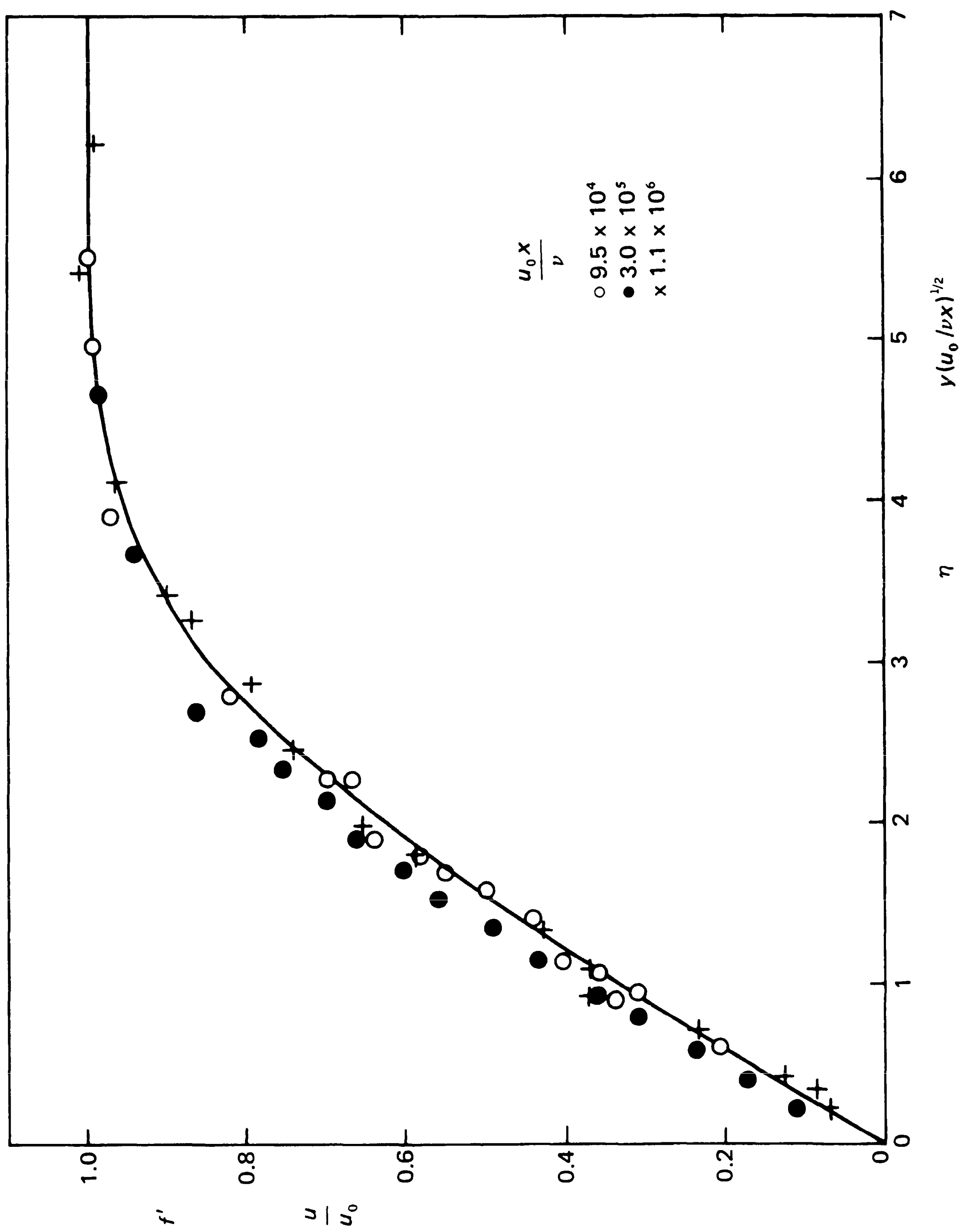

FIG. 11.2 Theoretical Blasius profile and experimental confirmation from Refs. [141] and [249].