They correlated the ratio of the settling velocity to the terminal velocity, Equation 4.68, with the volume fraction of the continuous phase. They also accounted for wall effects. For Reynolds numbers based on the terminal velocity between 200 and 500, the factor f works out to be

$$f = \alpha_c^{-k} \tag{4.81}$$

where $k = 4.45\,\mathrm{Re}^{-0.1}$ and $\mathrm{Re} = v_t D/\nu$.

Wen and Yu (1966) also conducted a series of fluidization experiments to infer the drag force on particles in dense mixtures. They were looking for a correction to the equation for drag force in the form

$$F_D = g(\alpha_c) 3\pi\mu_c D f_o (u - v) \tag{4.82}$$

where f_o is the drag factor for an isolated particle. Wen and Yu used the Schiller-Naumann correlation, Equation 4.51, in their analysis with the relative Reynolds number based on the superficial velocity. They were able to correlate their data and those of previous investigators (including Richardson and Zaki) by setting

$$g(\alpha_c) = \alpha_c^{-3.7} \tag{4.83}$$

The contribution of their analysis is that they included Reynolds number effects on the terminal velocity and, in so doing, were able to develop an empirical correlation over the entire Reynolds number regime. The drag factor f now becomes

$$f = \alpha_c^{-3.7} f_0 \tag{4.84}$$

As $\alpha_c \to 1$, $f \to f_0$.

Wen and Yu (1966) also claimed that their correlation provides the same results as Ergun's for volume fractions corresponding to minimum fluidization. This claim is somewhat dubious, however, since $g(\alpha_c)$ is so sensitive to α_c near minimum fluidization.

More recently, Di Felice (1994) found by analysis of various data available in the literature that

$$f = f_o \alpha_c^{-\beta} \tag{4.85}$$

where β is a function of the relative Reynolds number. In the low Reynolds number regime the value of β approaches 3.65 based on the data of Richardson and Zaki (1954). At high Reynolds numbers β approaches 3.7 from the data of Wen and Yu (1966) and others. In the intermediate range of Reynolds numbers β goes through a minimum value of approximately 3 for Reynolds numbers in the range 20 to 80. Di Felice recommends the following empirical correlation for β,[11]

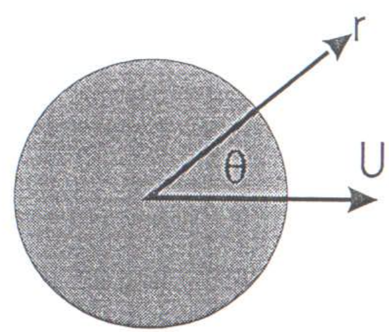

Figure 4.11: Coordinate system for sphere accelerating in a fluid.

$$\beta = 3.7 - 0.65 \exp\left[-\frac{(1.5 - \log Re_r)^2}{2}\right] \tag{4.86}$$

for relative Reynolds numbers from 10^{-2} to 10^4.

The present state of knowledge on the drag of particles in a cloud is still very primitive. A better understanding may be forthcoming through the current development of numerical models for arrays of particles (Dasgupta et al., 1994; Feng et al., 1994; Hu, 1996). However, the correlation proposed by Di Felice (1994) is recommended for the present time.

4.3.4 Unsteady forces

The forces due to acceleration of the relative velocity can be divided into two parts: the *virtual mass effect* and the *Basset force*. The virtual mass effect relates to the force required to accelerate the surrounding fluid. The Basset term describes the force due to the lagging boundary layer development with changing relative velocity.

Virtual or apparent mass effect

When a body is accelerated through a fluid, there is a corresponding acceleration of the fluid which is at the expense of work done by the body. This additional work relates to the virtual mass effect.

Consider a sphere in a fluid as shown in Figure 4.11. The total kinetic energy of the fluid surrounding the sphere is

$$KE = \frac{1}{2}\rho_c \int_V u^2 dV \tag{4.87}$$

where the integral is taken over all the fluid. It is assumed that the fluid is

inviscid & incompressible (ideal fluid) → the vel. can be expressed as the
derivative of a potential function

$$\mathbf{u} = \nabla\phi \tag{4.88}$$

The continuity equation in terms of the potential function is

$$\nabla \cdot \mathbf{u} = \nabla^2\phi = 0 \tag{4.89}$$

The kinetic energy of the fluid can be expressed in terms of the potential function as

$$KE = \frac{1}{2}\rho_c \int_V \nabla\phi \cdot \nabla\phi dV \tag{4.90}$$

However, because of the continuity equation, this integral can be written as

$$KE = \frac{1}{2}\rho_c \int_V [\nabla\phi\cdot\nabla\phi + \phi\nabla^2\phi]dV = \frac{1}{2}\rho_c \int_V \nabla\cdot(\phi\nabla\phi)dV \tag{4.91}$$

Using the divergence theorem, this volume integral can be expressed as a surface integral over the sphere (the sphere surface is the boundary "enclosing" the fluid)

$$KE = \frac{1}{2}\rho_c \int_V \nabla\cdot(\phi\nabla\phi)dV = \frac{1}{2}\rho_c \int_{cs} \phi\nabla\phi\cdot\mathbf{n}'dA \tag{4.92}$$

where $\mathbf{n}'$ is the unit outward normal vector from the fluid.

The potential function for a sphere moving with a relative velocity U through a fluid is

$$\phi = -\frac{Ua^3}{2r^2}cos\theta \tag{4.93}$$

where the angle θ is defined in the figure and a is the radius of the sphere. The radial component of velocity is

$$u_r = \frac{\partial\phi}{\partial r} = \frac{Ua^3}{r^3}cos\theta \tag{4.94}$$

which on the surface of the sphere reduces to

$$u_r = U\cos\theta \tag{4.95}$$

b

At $\theta = \pi$ the velocity u_r is $-U$ which is the velocity at that point in the radial direction. The dot product in Equation 4.92 becomes

$$\nabla\phi\cdot\mathbf{n}' = \frac{\partial\phi}{\partial r}\vec{e}_r\cdot(-\vec{e}_r) = -\frac{\partial\phi}{\partial r} = -U\cos\theta \tag{4.96}$$

ex

where $\vec{e}_r$ is the radial outward unit vector. Substituting the above expressions for ϕ and the gradient of ϕ into the equation for kinetic energy of the fluid gives

(*) $KE = \frac{1}{2}\rho_c \int_0^\pi \frac{Ua}{2} U\cos\theta\cos\theta a^2 \sin\theta\, 2\pi d\theta$

where $a^2 sin\theta 2\pi d\theta$ is the element of surface area on the sphere. This equation evaluates to

$$KE = \frac{\pi\rho_c a^3U^2}{2}\int_0^\pi cos^2\theta sin\theta d\theta = \frac{\pi\rho_c a^3 U^2}{3} \tag{4.98}$$

The work rate required to change the kinetic energy is

$$UF_{vm} = \frac{dKE}{dt} \tag{4.99}$$

where F_{vm} is the "virtual mass" force. Thus

$$UF_{vm} = \frac{2\pi\rho_c a^3}{3}U\frac{dU}{dt} \tag{4.100}$$

so the force is equal to

$$F_{vm} = \frac{M_f}{2}\frac{dU}{dt} \tag{4.101}$$

where M_f is the mass of fluid displaced by the sphere. The acceleration of the carrier fluid is the acceleration of a fluid element which is usually represented by the material derivative of the velocity, DU/Dt. This force, F_{vm} is the force of the particle on the fluid so the drag force is in the opposite sense. In general the relative acceleration of the fluid with respect to the particle acceleration is $\dot{\mathbf{u}} - \dot{\mathbf{v}}$ where $\dot{\mathbf{u}}$ is the material derivative of the velocity, $D\mathbf{u}/Dt$. If the fluid was at rest, then the virtual mass force on the particle should be in the direction opposite the particle acceleration. Thus the virtual mass force acting on the particle is given by

$$\boxed{\mathbf{F}_{vm}} = \frac{\rho_c V_d}{2}(\dot{\mathbf{u}} - \dot{\mathbf{v}}) = \boxed{\frac{\rho_c V_d}{2}(\frac{D\mathbf{u}}{Dt} - \frac{d\mathbf{v}}{dt})} \tag{4.102}$$

This force is sometimes called the *apparent mass* force because it is equivalent to adding a mass to the sphere. Analyses are available for shapes other than spheres for which the form of the equation is the same but the mass of fluid displaced is different.

Experiments for a sphere in simple harmonic motion (Odar & Hamilton, 1964) indicate that the virtual mass term also depends on the acceleration parameter which is defined as[12]

$$Ac = \frac{u_r^2}{D\frac{du_r}{dt}}$$

where u_r is the relative velocity. The acceleration parameter decreases as the relative velocity decreases or the relative acceleration increases. They proposed a coefficient to correct to the virtual mass term. Odar (1964) suggested the

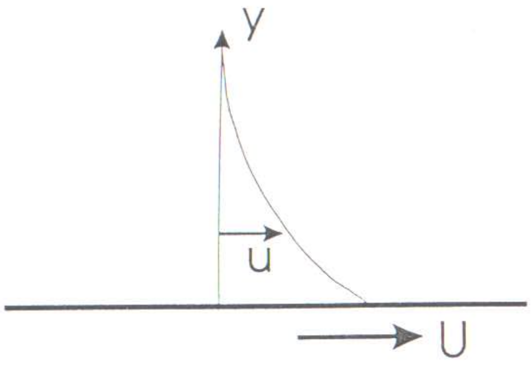

Figure 4.12: Impulsively accelerated flat plate.

following empirical equation for the coefficient, C_{vm}, as a function of the acceleration parameter,[13]

$$C_{vm} = 2.1 - \frac{0.132}{0.12 + Ac^2} \tag{4.103}$$

This correlation was developed from data using a sphere in simple harmonic motion. Subsequent work by Odar (1966) demonstrated the validity of the correlation for spheres dropping due to gravity in a tank of liquid. Further work by Schöneborn (1975) showed the utility of the correlation for predicting the fall velocity of particles in an tank of oscillating fluid.

Basset force

In that the virtual mass force accounts for the form drag due to acceleration, the Basset term accounts for the viscous effects. This term addresses the temporal delay in boundary layer development as the relative velocity changes with time. This term is sometimes called the "history" term.

The most direct approach to understanding the Basset force is to consider an impulsively accelerated infinite flat plate shown in Figure 4.12. The equation of motion for the fluid is

$$\frac{\partial u}{\partial t} = \nu_c \frac{\partial^2 u}{\partial y^2} \tag{4.104}$$

with the initial condition $u(0, y) = 0$ and the boundary conditions $u(t, 0) = u_0$ and $u(t, \infty) = 0$ where u_0 is the velocity of the plate. Thus the plate is started impulsively with a step change in velocity from 0 to u_0.

The solution to this equation is

$$u = u_0 \operatorname{erf}(\eta) \tag{4.105}$$

[13] The expression Odar proposed approaches 0.5 as $Ac \to 0$ which replaces the 0.5 factor in

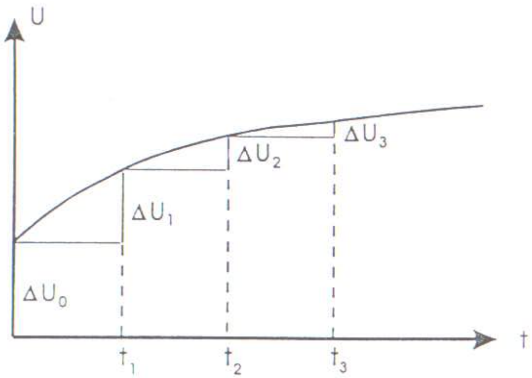

Figure 4.13: Stepwise impulsive acceleration of a flat plate.

where $\eta = \frac{y}{2\sqrt{\nu t}}$ or

$$u = \frac{2u_0}{\sqrt{\pi}} \int_0^{\eta} e^{-\lambda^2} d\lambda \tag{4.106}$$

The local shear stress is

$$\tau = \mu_c \frac{\partial u}{\partial y}|_{y=0} = \mu_c \frac{2u_0}{\sqrt{\pi}} \frac{\partial \eta}{\partial y} = \frac{\mu_c u_0}{\sqrt{\pi \nu_c t}} \tag{4.107}$$

or

$$\tau = \frac{\sqrt{\rho_c \mu_c} u_0}{\sqrt{\pi t}} \tag{4.108}$$

Now assume that a general temporal variation in plate velocity can be broken up into a series of step changes as shown in the Figure 4.13. At time 0 there is a change Δu_0, at time t_1 a change Δu_1 and so on. The cumulative effect on shear stress would be

$$\tau = \sqrt{\frac{\rho_c \mu_c}{\pi}} [\frac{\Delta u_o}{\sqrt{t}} + \frac{\Delta u_1}{\sqrt{t - t_1}} + \frac{\Delta u_2}{\sqrt{t - t_2}} ...] \tag{4.109}$$

For a time step $\Delta t'$ the change in velocity would be $\frac{du}{dt'}\Delta t'$ so the above sum can be expressed as

$$\tau = \sqrt{\frac{\rho_c \mu_c}{\pi}} \sum_{n=0}^{N} \frac{\frac{du}{dt'}}{\sqrt{t - n\Delta t'}} \Delta t' \tag{4.110}$$

(*) $\tau = \sqrt{\frac{\rho_c \mu_c}{\pi}} \int_0^t \frac{du/dt'}{\sqrt{t - t'}} dt'$

where $N\Delta t'$ represents the time interval from the initiation of the acceleration to the present time; that is, from 0 to t. In the limit as $\Delta t'$ approaches zero and $N\Delta t' \to t'$ the equation becomes

(*)

Applying this same approach to the impulsive flow over a sphere at low Reynolds number, Basset found that the drag force was equal to

$$\mathbf{F}_{Basset} = \frac{3}{2}D^2\sqrt{\pi\rho_c\mu_c}\int_0^t \frac{\dot{\mathbf{u}}-\dot{\mathbf{v}}}{\sqrt{t-t'}}dt' \tag{4.112}$$

where, once again, $\dot{\mathbf{u}}$ is the material derivative of the fluid velocity, $D\mathbf{u}/Dt$.The "historical" nature of this term is evident; the value of the Basset force depends on the acceleration history up to the present time. This term is often difficult to evaluate although important in many unsteady applications. According to the calculations of Hjemfelt and Mockros (1966), the Basset term and virtual mass term become insignificant for $\rho_c/\rho_d \sim 10^{-3}$ if $(\mu_c/\rho_c\omega D^2)^{1/2} > 6$ where ω is the frequency of the oscillating flow. Thus the Basset term would not be important for a 10-μm particle in a stream oscillating at less than 700 Hz (Rudinger, 1980). Voir and Michaelides (1994) have also shown that the Basset term is negligible for oscillatory velocity fields if $\rho_c/\rho_d < 0.002$ and $\omega\tau_V < 0.5$.

As with the virtual mass term, an empirical coefficient, C_B, has been proposed by Odar and Hamilton (1964) to account for the effect of acceleration on the Basset term. The coefficient as given by Odar (1966) is[14]

$$C_B = 0.48 + \frac{0.52}{(1+Ac)^3} \tag{4.113}$$

Reeks and McKee (1984) have shown that the Basset term has to be modified to include the case when there is an initial velocity. The term becomes

$$\mathbf{F}_{Basset} = \frac{3}{2}D^2\sqrt{\pi\rho_c\mu_c}\left[\int_0^t \frac{\dot{\mathbf{u}}-\dot{\mathbf{v}}}{\sqrt{t-t'}}dt' + \frac{(\mathbf{u}-\mathbf{v})_0}{\sqrt{t}}\right] \tag{4.114}$$

where $(\mathbf{u}-\mathbf{v})_0$ is the initial velocity difference. Mei et al. (1991) developed a numerical model for stationary flow over a sphere with small free-stream velocity fluctuations. They found that the unsteady Stokes equation does not describe the character of unsteady drag at low frequencies and suggested that this effect may explain the observations of McKee and Reeks that the initial velocity difference has a finite contribution to the long-term particle diffusivity in a turbulent flow.

4.3.5 Basset-Boussinesq-Oseen equation

Equating the sum of the steady-state drag force, the pressure (buoyancy) force, virtual mass force, the Basset force and the body force to the mass times the acceleration of an isolated droplet or particle yields the Basset-Boussinesq-Oseen

(BBO) equation for particle or droplet motion.[15]

$$\begin{aligned} m\frac{d\mathbf{v}}{dt} &= 3\pi\mu_c D(\mathbf{u}-\mathbf{v}) + V_d(-\nabla p + \nabla\cdot\tau) + \frac{\rho_c V_d}{2}(\dot{\mathbf{u}}-\dot{\mathbf{v}}) \\ &+\frac{3}{2}D^2\sqrt{\pi\rho_c\mu_c}\left[\int_0^t \frac{\dot{\mathbf{u}}-\dot{\mathbf{v}}}{\sqrt{t-t'}}dt' + \frac{(\mathbf{u}-\mathbf{v})_0}{\sqrt{t}}\right] + m\mathbf{g} \end{aligned} \tag{4.115}$$

A rigorous derivation of the equation of motion of small particles in nonuniform flows has been performed by Maxey and Riley (1983). The equation they derive is essentially the same as the one above except for extra terms due to the nonuniformity of the velocity field and the additional term in the Basset force. If nonuniformity effects are included the Faxen force appears in the steady-state drag term (Equation 4.46). Also the virtual mass term becomes

$$\frac{\rho_c V_d}{2}(\dot{\mathbf{u}} - \dot{\mathbf{v}} - \frac{D^2}{40}\frac{d}{dt}\nabla^2\mathbf{u}) \tag{4.116}$$

and an additional term appears in the Basset force. For the sake of simplicity, the terms due to flow field nonuniformity will not be included here.

Dividing through Equation 4.115 by the droplet mass and rearranging the virtual mass term gives

$$\begin{aligned} \left(1+\frac{1}{2}\frac{\rho_c}{\rho_d}\right)\frac{d\mathbf{v}}{dt} &= \frac{1}{\tau_V}(\mathbf{u}-\mathbf{v}) + \frac{1}{\rho_d}(-\nabla p + \nabla\tau_i) + \frac{1}{2}\frac{\rho_c}{\rho_d}\dot{\mathbf{u}} \\ &+\sqrt{\frac{9}{2\pi}}\left(\frac{\rho_c}{\rho_d}\right)^{\frac{1}{2}}\frac{1}{\sqrt{\tau_V}}\left[\int_0^t \frac{\dot{\mathbf{u}}-\dot{\mathbf{v}}}{\sqrt{t-t'}}dt' + \frac{(\mathbf{u}-\mathbf{v})_0}{\sqrt{t}}\right] + \mathbf{g} \end{aligned} \tag{4.117}$$

The pressure gradient and shear stress term can be related to the fluid acceleration and force due to gravity from the Navier-Stokes equation for the conveying fluid

$$-\nabla p + \nabla\cdot\tau_{ij} = \rho_c\frac{D\mathbf{u}}{Dt} - \rho_c\mathbf{g}$$

so the combination of the pressure gradient and shear stress term can be combined with the fluid acceleration in the virtual mass term to yield the following form of the BBO equation[16]

$$\begin{aligned} \left(1+\frac{1}{2}\frac{\rho_c}{\rho_d}\right)\frac{d\mathbf{v}}{dt} &= \frac{1}{\tau_V}(\mathbf{u}-\mathbf{v}) + \frac{3}{2}\frac{\rho_c}{\rho_d}\dot{\mathbf{u}} \\ &+\sqrt{\frac{9}{2\pi}}\left(\frac{\rho_c}{\rho_d}\right)^{\frac{1}{2}}\frac{1}{\sqrt{\tau_V}}\left[\int_0^t \frac{\dot{\mathbf{u}}-\dot{\mathbf{v}}}{\sqrt{t-t'}}dt' + \frac{(\mathbf{u}-\mathbf{v})_0}{\sqrt{t}}\right] + \mathbf{g}\left(1-\frac{\rho_c}{\rho_d}\right) \end{aligned} \tag{4.118}$$

For flows, such as gas-particle flows, where the ratio of the continuous phase density to the droplet material density is very small ($\sim 10^{-3}$), the BBO equation can be justifiably simplified to

$$\frac{d\mathbf{v}}{dt} = \frac{1}{\tau_V}(\mathbf{u}-\mathbf{v}) + \mathbf{g} \tag{4.119}$$

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[15] The Faxen force is not included here.