PARTICLE TRACKING IN A CELLULAR FLOW FIELD

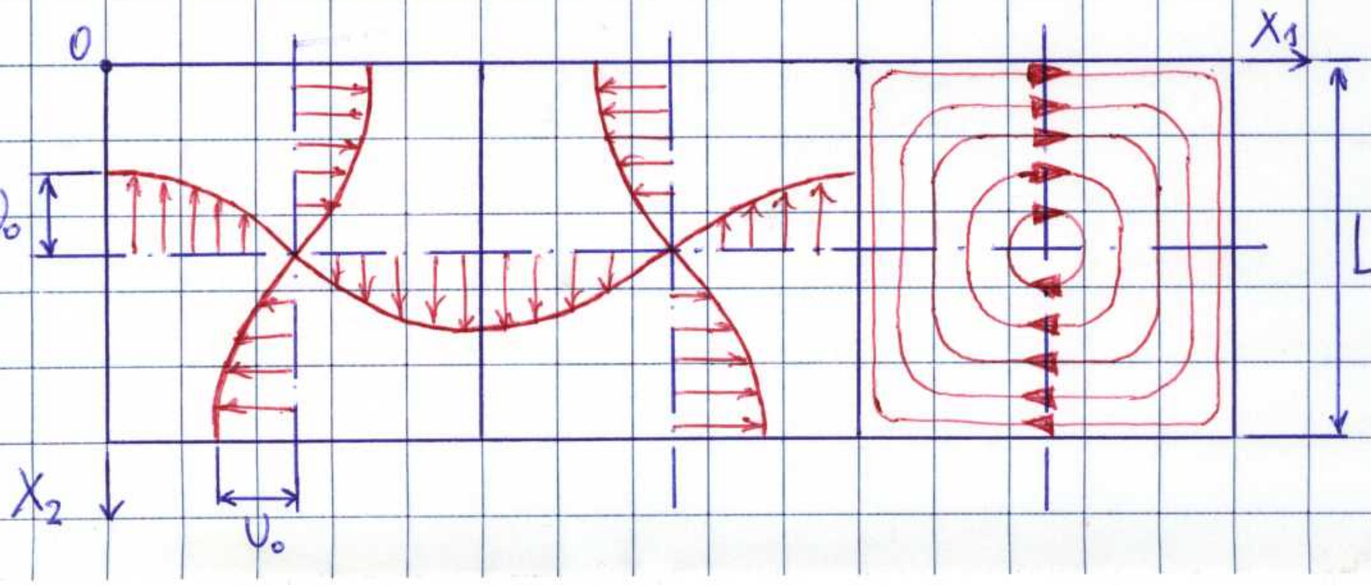
Objective : simulate numerically the dispersion of small particles in a simplified turbulent flow field

(A) FLOW FIELD

2D periodic flow of cellular vortices

Incompressible

$$\psi = U_0 L \sin(X_1/L) \sin(X_2/L)$$



Fluid velocity components :

$$u_1 = \frac{dX_1}{dt} = U_0 \sin(X_1/L)\cos(X_2/L)$$

$$u_2 = \frac{dX_2}{dt} = -U_0 \cos(X_1/L)\sin(X_2/L)$$

(B) PARTICLE MOTION EQUATIONS

Start from Maxey & Riley (1983):

$$\underbrace{m_p \frac{d\vec{V}}{dt}}_{\text{INERTIA}} = \underbrace{(m_p - m_f)\vec{g}}_{\text{BUOYANCY}} + \underbrace{m_f \left.\frac{D\vec{u}}{Dt}\right|_{\vec{Y}(t)}}_{\text{EFFECTS OF PRESSURE GRAD. OF THE UNDISTURBED FLOW}}$$

$\frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u}$

$\frac{d\vec{u}}{dt} = \frac{\partial \vec{u}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{u}$

$$\underbrace{-\frac{1}{2} m_f \frac{d}{dt}\left[\vec{V} - \vec{u}\big|_{\vec{Y}(t)}\right]}_{\text{ADDED MASS}} \underbrace{- 6\pi a \mu \left[\vec{V} - \vec{u}\big|_{\vec{Y}(t)}\right]}_{\text{STOKES DRAG}}$$

$\vec{X} \triangleq \left[\vec{V} - \vec{u}\big|_{\vec{Y}(t)}\right]$

$$\underbrace{- 6\pi a^2 \mu \int_0^t \frac{d\vec{X}/d\tau}{\sqrt{\pi\nu(t-\tau)}}\, d\tau}_{\text{BASSET HISTORY TERM}} + \underbrace{6{,}46\, \rho_f\, a^2 \sqrt{\nu} \cdot \Gamma}_{\text{LIFT}}$$

$m_p = \rho_p \cdot \frac{4}{3}\pi a^3$ MASS OF THE PARTICLE

$\hookrightarrow$ a = PARTICLE RADIUS

$m_f = \rho_f \cdot \frac{4}{3}\pi a^3$ MASS OF FLUID DISPLACED BY THE PARTICLE

$\vec{V} \triangleq \vec{V}(t) \triangleq V_1 \vec{i} + V_2 \vec{j}$ PARTICLE VELOCITY

$\vec{u} \triangleq u_1 \vec{i} + u_2 \vec{j}$ FLUID VELOCITY

$\vec{Y}(t) \triangleq Y_1(t)\vec{i} + Y_2(t)\vec{j}$ PARTICLE POSITION

$$\Gamma \triangleq \frac{\left[(\vec{u} - \vec{V}) \times (\vec{\nabla} \times \vec{u})\right]}{\sqrt{|\vec{\nabla} \times \vec{u}|}}$$ with $\vec{\nabla} \times \vec{u} \triangleq$ VORTICITY

Neglecting Basset and re-arranging:

$$\frac{d\vec{V}}{dt} = \frac{m_p - m_f}{m_p + \frac{1}{2}m_f}\vec{g} + \frac{m_f}{m_p + \frac{1}{2}m_f}\vec{u} \cdot \vec{\nabla}\vec{u}$$

$$+ \frac{1}{2}\frac{m_f}{m_p + \frac{1}{2}m_f}\vec{V} \cdot \vec{\nabla}\vec{u} - \frac{6\pi a \mu (\vec{V} - \vec{u})}{m_p + \frac{1}{2}m_f}$$

$$+ \frac{6,46\,\rho_f\, a^2 \sqrt{\nu}}{m_p + \frac{1}{2}m_f} \cdot \Gamma$$

$$② \quad \frac{W^{(s)}}{Q} = \left(\frac{m_p - m_f}{6\pi a \mu} \cdot \frac{1}{U_0} \right) \cdot \frac{1}{\left(\frac{m_f}{6\pi a \mu} \cdot \frac{1}{U_0} \right)} =$$

$$= \frac{m_p - m_f}{m_f} = \frac{m_p + \frac{1}{2} m_f - \frac{3}{2} m_f}{m_f} = \frac{1}{R} - \frac{3}{2} \quad \boxed{W^{(s)} = \left(\frac{1}{R} - \frac{3}{2}\right) Q}$$

For a bubble:

$$\rho_p << \rho_f \Rightarrow m_p << m_f \rightarrow \vec{W}^{(s)} \cong - \frac{m_f \vec{g}}{6\pi a \mu}$$

Define $\boxed{Q = |\vec{W}^{(s)*}| = \frac{|\vec{W}^{(s)}|}{U_0} = \frac{m_f \vec{g}}{6\pi a \mu} \cdot \frac{1}{U_0}}$

Similarly: $A \cong \boxed{\frac{6\pi a \mu}{\frac{1}{2} m_f} \cdot \frac{L}{U_0} \equiv B}$

Therefore $Q \equiv |\vec{W}^{(s)*}|$ and $B = A$ in the limit $\rho_p \rightarrow 0$.

Find: ① $\frac{A}{B} = \left(\frac{6\pi a \mu}{m_p + \frac{1}{2} m_f} \cdot \frac{L}{U_0} \right) \cdot \frac{1}{\left(\frac{6\pi a \mu}{\frac{1}{2} m_f} \cdot \frac{L}{U_0} \right)}$

$$= \frac{1}{2} \frac{m_f}{m_p + \frac{1}{2} m_f} = \frac{1}{2} R \rightarrow \boxed{A = \frac{BR}{2}}$$

Define :

- $0 < R \leq 0.4$ AEROSOL RANGE
- $0.4 < R < 2$ TRANSITION RANGE
- $R = 2$ BUBBLE RANGE

- $R \triangleq \dfrac{m_f}{m_p + \frac{1}{2} m_f}$ MASS RATIO

$$\Rightarrow \text{if } \rho_p \gg \rho_f \text{ then } R \to 0$$

$$\Rightarrow \text{if } \rho_p \ll \rho_f \text{ then } R \to 2$$

- $\alpha \triangleq \dfrac{6 \pi a \mu}{m_p + \frac{1}{2} m_f}$ INERTIA PARAMETER

$$\tau_p = \frac{(\rho_p + \frac{1}{2}\rho_f) D_p^2}{18 \mu} = \frac{(\rho_p + \frac{1}{2}\rho_f) \frac{\pi D_p^3}{6}}{18 \mu} \cdot \frac{6}{\pi D_p} = \frac{m_p + \frac{1}{2} m_f}{3 \pi \mu D_p} = \frac{m_p + \frac{1}{2} m_f}{6 \pi a \mu} = \frac{1}{\alpha} \; !!$$

- $\vec{W}^{(s)} \triangleq \dfrac{m_p - m_f}{6 \pi a \mu} \vec{g}$ STOKES SETTLING VEL.

$$\Rightarrow \vec{W}^{(s)} = \frac{m_p - m_f}{6 \pi a \mu} \vec{g} \cdot \frac{m_p + \frac{1}{2} m_f}{m_p + \frac{1}{2} m_f} = \frac{m_p - m_f}{m_p + \frac{1}{2} m_f} \vec{g} \cdot \frac{1}{\alpha}$$

- $\mathcal{L} \triangleq \dfrac{6{,}46 \, \rho_f \, a^2 \sqrt{\nu}}{m_p + \frac{1}{2} m_f} = \ldots = \dfrac{1{,}5422}{a} \sqrt{\dfrac{\mu}{\rho_f}} \cdot R$

to get:

$$\frac{d\vec{V}}{dt} = \alpha \vec{W}^{(s)} + R\left(\vec{u}\cdot\vec{\nabla}\vec{u} - \frac{1}{2}\vec{V}\cdot\nabla\vec{u}\right) - \alpha\left(\vec{V}-\vec{u}\right) + \mathcal{L}\cdot\Gamma \qquad (1)$$

Equation in non-dimensional form:

take $\vec{V}^* = \vec{V}/U_0$

$\vec{u}^* = \vec{u}/U_0$

$\vec{W}^{(s)*} = \vec{W}^s/U_0$

$\vec{X}^* = \vec{X}/L$ + $A = \dfrac{\alpha L}{U_0}$

$\vec{Y}^* = \vec{Y}/L$

$t^* = t\,.\,U_0/L$

substitute into (1) and re-arrange to obtain:

$$\frac{d\vec{V}^*}{dt^*} = A\left[\vec{W}^{(s)*} + \left(\vec{u}^* - \vec{V}^*\right)\right] + R\left[\vec{u}^*\cdot\vec{\nabla}^*\vec{u}^* + \frac{1}{2}\vec{V}^*\cdot\vec{\nabla}^*\vec{u}^*\right] + 0{,}727\sqrt{A\cdot R}\,.\,\frac{\left[\left(\vec{u}^*-\vec{V}^*\right)\times\left(\vec{\nabla}^*\times\vec{u}^*\right)\right]}{\sqrt{\left|\vec{\nabla}^*\times\vec{u}^*\right|}} \qquad (2)$$

Dropping apex *, Eq. (2) in scalar form reads as:

$$\frac{dV_i}{dt} = A(u_i - V_i) + R\left[u_j \frac{\partial u_i}{\partial x_j} + \frac{1}{2} V_j \frac{\partial u_i}{\partial x_j}\right]$$

$$+ \frac{0{,}727\sqrt{AR}\,\left[(\vec{u} - \vec{V}) \times (\vec{\nabla} \times \vec{u})\right]\Big|_i}{\sqrt{|\vec{\nabla} \times \vec{u}|}\Big|_i}$$

where $i = 1, 2$.

Note:

$$(\vec{u} - \vec{V}) \times (\vec{\nabla} \times \vec{u}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 - V_1 & u_2 - V_2 & 0 \\ 0 & 0 & \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \end{vmatrix} =$$

$$= \vec{i}\left[(u_2 - V_2)\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\right] - \vec{j}\left[(u_1 - V_1)\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\right] =$$

$$= \vec{i}\left[2 \sin Y_1 \sin Y_2 \left(-\cos Y_1 \sin Y_2 - V_2\right)\right] + \vec{j}\left[2 \sin Y_1 \sin Y_2 \left(V_1 - \sin Y_1 \cdot \cos Y_2\right)\right]$$

In summary, the epuations to be integrated in time for tracking particles in the cellular flow are:

$$\frac{1}{A}\frac{dV_1}{dt} + V_1 = \sin Y_1 \cos Y_2$$
$$+ \frac{1}{2}\frac{R}{A}\left(V_1 \cos Y_1 \cos Y_2 - V_2 \sin Y_1 \sin Y_2\right)$$
$$+ \frac{R}{A} \sin Y_1 \cos Y_2$$
$$+ 0{,}727\sqrt{\frac{R}{A}} \cdot \frac{(V_2 - \cos Y_1 \sin Y_2)(-2\sin Y_1 \sin Y_2)}{\sqrt{2\,|\sin Y_1 \sin Y_2|}}$$

$$\frac{1}{A}\frac{dV_2}{dt} + V_2 = -\cos Y_1 \sin Y_2$$
$$+ \frac{1}{2}\frac{R}{A}\left(V_1 \sin Y_1 \sin Y_2 - V_2 \cos Y_1 \cos Y_2\right)$$
$$+ \frac{R}{A} \sin Y_1 \cos Y_2$$
$$+ 0{,}727\sqrt{\frac{R}{A}} \cdot \frac{(V_1 - \sin Y_1 \cos Y_2) \sin Y_1 \sin Y_2}{\sqrt{2\,|\sin Y_1 \sin Y_2|}}$$

Possible projects:

1. Compute average settling/rising velocity $\langle V_2 \rangle$ and compare against Stokes velocity $W^{(s)}$

2. Calculate equilibrium points and perform linear stability analysis $\Longrightarrow$

3. Compute the compressibility of the particle phase $\longrightarrow$ Jacobian

Do this considering different values of the particle-to-fluid density ratio ρ_p/ρ_f and different sets of forces acting on the particles (I: M&R87; II: M&R87 + Added Mass; III: M&R87 + Lift + Added Mass)

Technical details :

- INITIAL & BOUNDARY CONDITION

⇒ Initial particle position : random distr.

⇒ Initial particle velocity :

× equal to zero ($V_1 = V_2 = 0$ @ $t=0$)

× equal to the fluid velocity at the particle initial position

($V_i = u_i|_{Y_i(t)}$ @ $t=0$)

⇒ Boundary condition : periodic

- TIME INTEGRATION

⇒ Runge - Kutta (4^{th} - order)

× subroutine available at

www.nr.com

$\Longrightarrow$ Solve for the following system of 1^{st}-order ODEs :

$$
\begin{cases}
\dfrac{dY_1(t)}{dt} = V_1(t) \\[2ex]
\dfrac{dY_2(t)}{dt} = V_2(t) \\[2ex]
\dfrac{dV_1(t)}{dt} = A \cdot \left[-V_1 + \sin Y_1 \cos Y_2 + \dots \right] \\[2ex]
\dfrac{dV_2(t)}{dt} = A \cdot \left[-V_2 - \cos Y_1 \sin Y_2 + \dots \right]
\end{cases}
$$