VORTICITY DYNAMICS

Definition of vorticity:

1. MATHEMATICAL DEFINITION : $\boxed{\vec{\omega} = \vec{\nabla} \times \vec{v}}$

Vorticity is the curl of the fluid velocity. In cartesian coordinates:

$$\vec{\omega} = \vec{\nabla} \times \vec{v} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial \cdot}{\partial x} & \frac{\partial \cdot}{\partial y} & \frac{\partial \cdot}{\partial z} \\ v_x & v_y & v_z \end{pmatrix}$$

$$= \vec{i} \underbrace{\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right)}_{\omega_x} + \vec{j} \underbrace{\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right)}_{\omega_y} + \vec{k} \underbrace{\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right)}_{\omega_z}$$

According to this definition, each vorticity component $\omega_k \triangleq \frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}$ can be related to the fluid velocity gradient tensor:

$$\vec{\nabla} \vec{v} = \frac{\partial v_j}{\partial x_i} = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_x}{\partial z} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} \\ \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} & \frac{\partial v_z}{\partial z} \end{pmatrix}$$

Like any other tensor, $\nabla \vec{v}$ can be decomposed into the sum of a symmetric part:

$$\vec{S} \triangleq \frac{1}{2}\left(\overline{\nabla} \vec{v} + \overline{\nabla} \vec{v}^{T}\right)$$

$$S_{ij} \triangleq \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

called <u>STRAIN-RATE TENSOR</u>, and an anti-symmetric part:

$$\vec{\Omega} \triangleq \frac{1}{2}\left(\overline{\nabla} \vec{v} - \overline{\nabla} \vec{v}^{T}\right)$$

$$\Omega_{ij} \triangleq \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$$

called <u>ROTATION-RATE TENSOR</u>. Clearly: $\nabla \vec{v} = \vec{\Omega} + \vec{S}$

It is easy to show that:

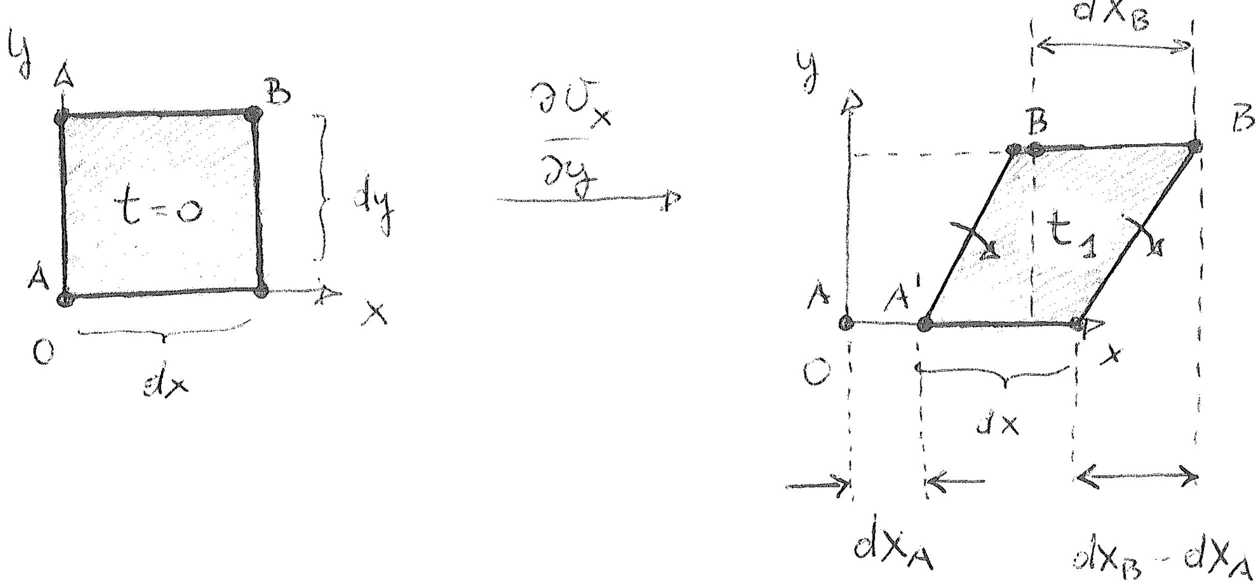
$$\vec{\omega} = -2\vec{\Omega} \quad \Rightarrow \quad \omega_k = -2\Omega_{ij}$$

where $i, j, k = x, y, z$ or y, z, x or z, x, y.

Hence, vorticity is twice the fluid's rotation rate.

2. PHYSICAL MEANING: consider a 2D fluid element, with side lengths $dx = dy$ subject to both velocity gradients $\partial v_x/\partial y$ and $\partial v_y/\partial x$.

Such element will thus be subject to ω_z.
Let us examine the effect of these two gradients on the fluid element:



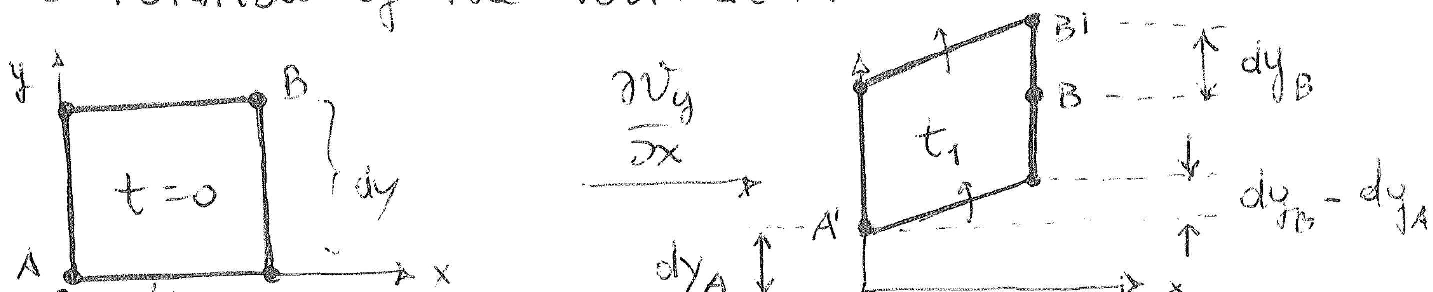
where: $dX_A = v_x^A \cdot dt$

$$dX_B = v_x^B \cdot dt \cong \left(v_x^A + \frac{\partial v_x}{\partial y}\, dy\right) dt = dX_A + \frac{\partial v_x}{\partial y}\, dy \ldots$$

$$\frac{\partial v_x}{\partial y} = \lim_{dy \to 0} \frac{v_x^B - v_x^A}{dy}$$

Therefore: $\boxed{dX_B - dX_A = \frac{\partial v_x}{\partial y}\, dy\, dt}$

and the fluid element is deformed as a result of a rotation of the vertical sides. Similarly:

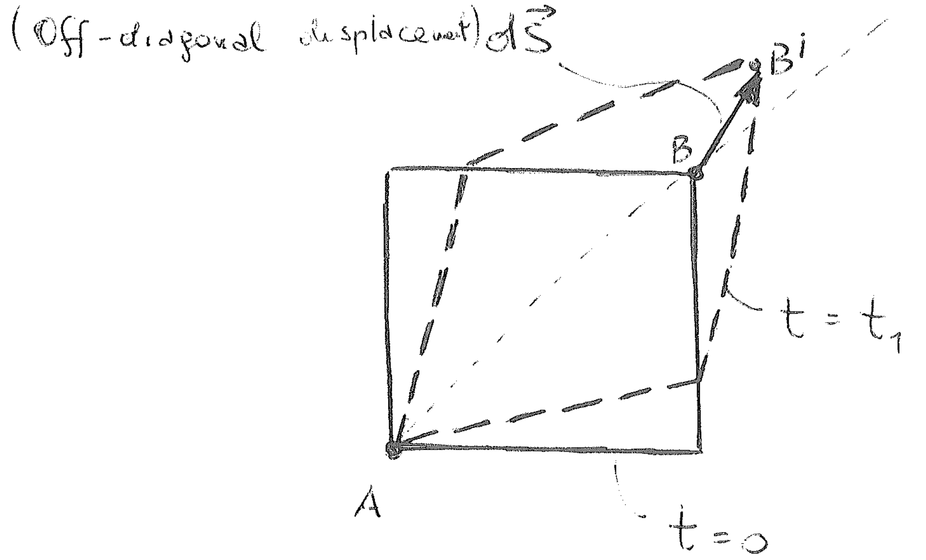


where: $dy_A = v_y^A \cdot dt$

$$dy_B = v_y^B \cdot dt \cong \left(v_y^A + \frac{\partial v_y}{\partial x} dx \right) dt$$

$$\frac{\partial v_y}{\partial x} = \lim_{dx \to 0} \frac{v_y^B - v_y^A}{dx}$$

Therefore: $dy_B - dy_A = \dfrac{\partial v_y}{\partial x} dx \, dt$



TOTAL DEFORMATION OF THE FLUID ELEMENT SUBJECT TO BOTH $\dfrac{\partial v_x}{\partial y}$ AND $\dfrac{\partial v_y}{\partial x}$

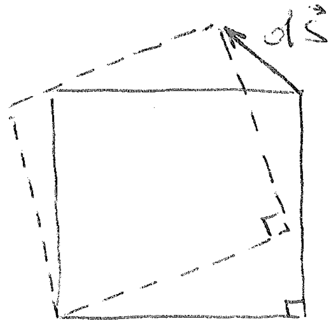
Displacement: $d\vec{S} \triangleq \left(\dfrac{\partial v_x}{\partial y} dy \, \vec{i} + \dfrac{\partial v_y}{\partial x} dx \, \vec{j} \right) dt$

Vorticity quantifies the total deformation of a fluid element that is subject to both translation and rotation.

There are two special cases:

1. $\boxed{\dfrac{\partial v_x}{\partial y} = - \dfrac{\partial v_y}{\partial x}} \longrightarrow \omega_z = 2 \dfrac{\partial v_y}{\partial x} = 2 \Omega_{yx}$

In this case, the fluid element rotates like a solid body without deforming (it remains square):

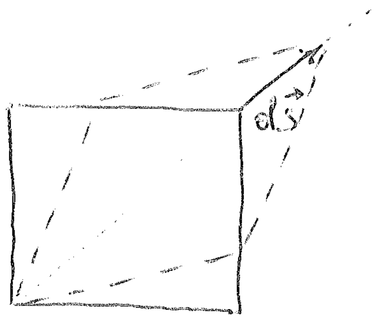


PURE ROTATION OF A FLUID ELEMENT (2D)

There is no deformation (and $S_{ij} = 0$!)

2. $\boxed{\dfrac{\partial v_x}{\partial y} = + \dfrac{\partial v_y}{\partial x}} \Longrightarrow \omega_z = 0 \quad (\Omega_{ij} = 0)$

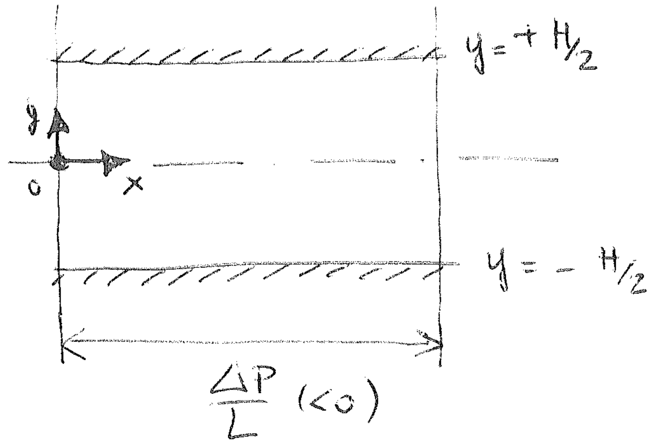
In this case, the fluid element does not rotate but it deforms as follows:



PURE STRAIN OF A FLUID ELEMENT (2D)

The diagonal of the fluid element keeps its angular orientation (and $S_{ij} \neq 0$!)

Example: calculation of vorticity in plane Poiseuille flow (in a channel)



$$v_x(y) = \frac{1}{2\mu}\left(\frac{\Delta P}{L}\right)\left(y^2 - \frac{H^2}{4}\right)$$

$$\omega_x = \omega_y = 0$$

$$\omega_z = \underbrace{\frac{\partial v_y}{\partial x}}_{=0} - \frac{\partial v_x}{\partial y}$$

$$= -\frac{1}{\mu}\left(\frac{\Delta P}{L}\right) y$$

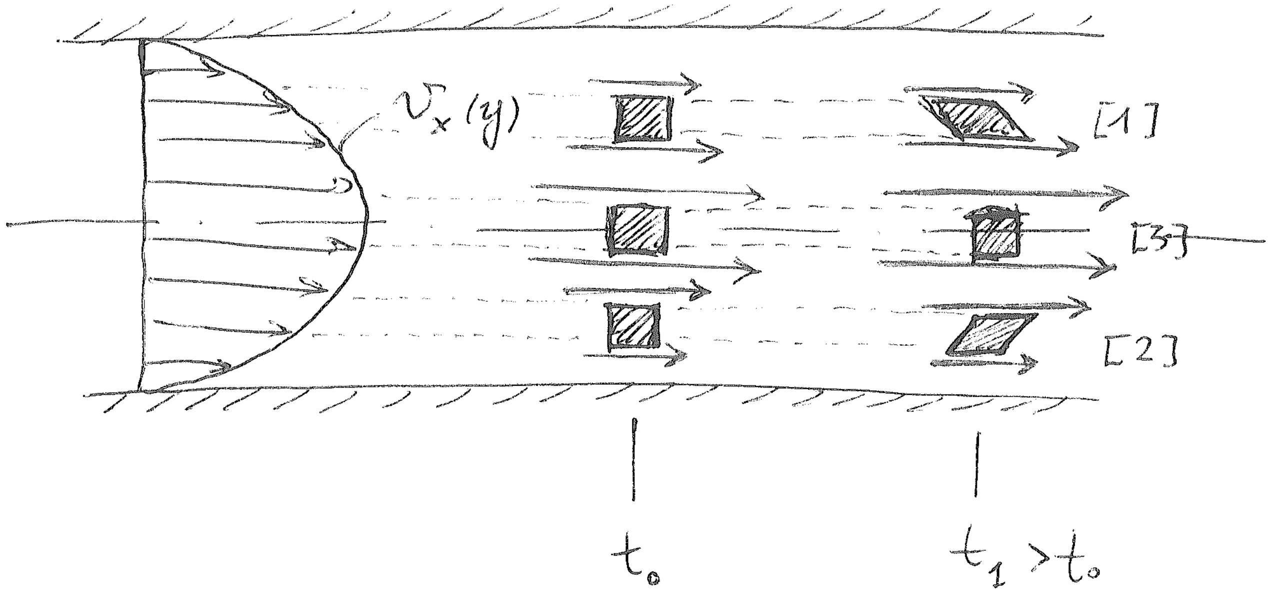
NOTE : $\tau_{yx} = \tau_{xy} = \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) = -\mu \cdot \omega_z$!

At the upper wall : $\omega_z(y = +H/2) = -\frac{1}{2\mu}\left(\frac{\Delta P}{L}\right) H > 0$

At the channel center : $\omega_z(y = 0) = 0$

At the lower wall : $\omega_z(y = -H/2) = \frac{1}{2\mu}\left(\frac{\Delta P}{L}\right) H < 0$

Vorticity is positive in the upper half of the channel (maximum at the upper wall) and negative in the lower half of the channel (maximum, in absolute value, at the lower wall), while vanishing at the centerline. This is due to the velocity profile, and to the vel. gradient $\partial v_x / \partial y$ in particular :



Element [1] moves in the streamwise direction experiencing a deformation due to the higher velocity with which its bottom side moves with respect to the upper side (which is "slower").
This deformation is related to a counter-clockwise rotation of the vertical sides, which corresponds to positive vorticity.
Element [2] moves in the streamwise direction experiencing a deformation due to the higher velocity with which its upper side moves. This leads to a clockwise rotation of the vertical side (namely to negative vorticity).
Element [3] just translates downstream without deforming (due to the symmetry of the velocity profile): no rotation occurs, and therefore $\omega_z = 0$.

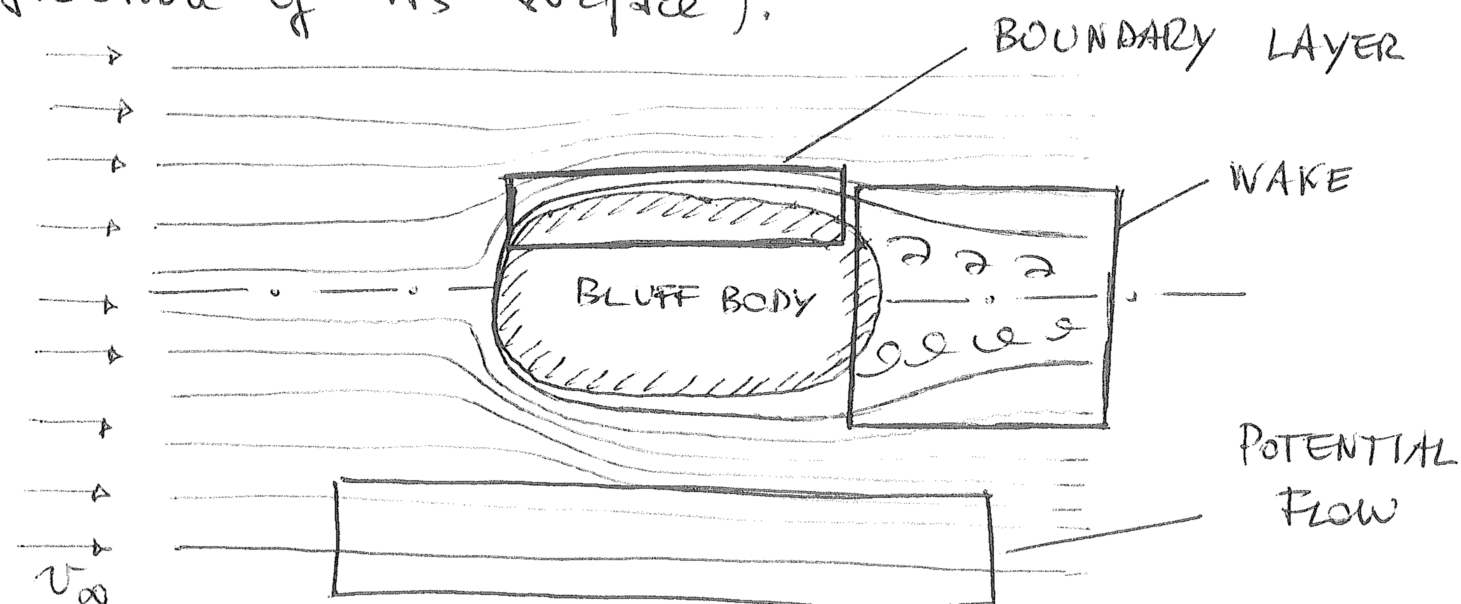
Note that in such a simple flow:

$$\nabla\vec{v} = \begin{pmatrix} 0 & \frac{\partial \bar{v}_x}{\partial y} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \vec{\Omega} = \frac{1}{2}\begin{pmatrix} 0 & \frac{\partial \bar{v}_x}{\partial y} & 0 \\ -\frac{\partial \bar{v}_x}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\omega_z = 2\,\Omega_{xy} = 2\,S_{xy}\,! \qquad \vec{S} = \frac{1}{2}\begin{pmatrix} 0 & \frac{\partial \bar{v}_x}{\partial y} & 0 \\ \frac{\partial \bar{v}_x}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

— ∘ —

Vorticity can be used to discriminate between regions of turbulent flows in which the fluid exhibits different behaviour. A rough classification of these regions can be given considering the flow of a fluid around a bluff body (namely a body that is not streamlined and gives rise to flow separation over a non-negligible fraction of its surface):



We can identify 3 distinct flow regions:

1. <u>BOUNDARY LAYER</u> (close to the surface of the bluff body): in this thin region of the flow, vorticity is different from zero and viscous dissipation ($\alpha \frac{\partial v_i}{\partial x_j}$) is important.

2. <u>WAKE</u> (downstream of the bluff body): in this region, which is generated by separation of the boundary layer from the surface of the bluff body, vorticity is different from zero (this typically accounts for vortex shedding phenomena such as the Von Karman vortex street) but viscous dissipation becomes negligible.

3. <u>POTENTIAL FLOW</u> (far enough away from the bluff body): in this region of the flow, the motion of the fluid is practically undisturbed by the presence of the bluff body. Therefore the streamlines are not deformed, vorticity vanishes and viscous dissipation is also negligible.

VORTICITY TRANSPORT EQUATION

The time- and space- evolution of vorticity in a fluid flow is described by a transport equation that resembles the momentum conservation equation. Indeed, this equation can be obtained starting from the Navier-Stokes equation:

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\bar{\nabla}\vec{v}\right) = -\bar{\nabla}P + \mu\bar{\nabla}^2\vec{v}$$

DIVIDE BY ρ

$$\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\bar{\nabla}\vec{v} = -\frac{1}{\rho}\bar{\nabla}P + \nu\bar{\nabla}^2\vec{v}$$

TAKE THE CURL

$$\bar{\nabla}\times\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\bar{\nabla}\vec{v}\right) = \bar{\nabla}\times\left(-\frac{1}{\rho}\bar{\nabla}P + \nu\bar{\nabla}^2\vec{v}\right)$$

$$\bar{\nabla}\times\frac{\partial \vec{v}}{\partial t} + \bar{\nabla}\times(\vec{v}\cdot\bar{\nabla}\vec{v}) \qquad -\bar{\nabla}\times\left(\frac{\bar{\nabla}P}{\rho}\right) + \nu\bar{\nabla}\times\bar{\nabla}^2\vec{v}$$

Due to the linearity of all operators, one gets:

- $\bar{\nabla}\times\dfrac{\partial \vec{v}}{\partial t} = \dfrac{\partial}{\partial t}(\bar{\nabla}\times\vec{v}) = \dfrac{\partial \vec{\omega}}{\partial t}$
- $\bar{\nabla}\times(\vec{v}\cdot\bar{\nabla}\vec{v}) = \bar{\nabla}\times\left[\bar{\nabla}\left(\frac{1}{2}\vec{v}\cdot\vec{v}\right) - \vec{v}\times\vec{\omega}\right]$

$$= \underbrace{\cancel{\bar{\nabla}\times\bar{\nabla}\left(\frac{1}{2}v^2\right)}}_{=0} - \bar{\nabla}\times(\vec{v}\times\vec{\omega})$$

- $$-\overline{\nabla}\times\left(\frac{\overline{\nabla}P}{\rho}\right) = -\left[\overline{\nabla}\left(\frac{1}{\rho}\right)\times\overline{\nabla}P + \underbrace{\frac{1}{\rho}\cancel{\overline{\nabla}\times\overline{\nabla}P}}_{\text{CURL(GRAD(P))}=0}\right]$$

$$= -\left[-\frac{1}{\rho^2}\overline{\nabla}\rho\times\overline{\nabla}P\right] = \frac{\overline{\nabla}\rho\times\overline{\nabla}P}{\rho^2}$$

- $$\nu\,\overline{\nabla}\times\overline{\nabla}^2\vec{v} = \nu\,\overline{\nabla}^2(\overline{\nabla}\times\vec{v}) = \nu\,\overline{\nabla}^2\vec{\omega}$$

Namely:

$$[\star]\quad \underbrace{\frac{\partial\vec{\omega}}{\partial t}}_{\substack{\text{ACCUMULATION}\\ \text{TERM (CAN BE}\\ >0 \text{ OR } <0 !)}} - \overline{\nabla}\times(\vec{v}\times\vec{\omega}) = \underbrace{\frac{\overline{\nabla}\rho\times\overline{\nabla}P}{\rho^2}}_{\substack{\text{BAROCLINIC}\\ \text{TERM}}} + \underbrace{\nu\,\overline{\nabla}^2\vec{\omega}}_{\substack{\text{DIFFUSION OF}\\ \text{VORTICITY}}}$$

Now: $$\overline{\nabla}\times(\vec{v}\times\vec{\omega}) = \underbrace{\cancel{\vec{v}(\overline{\nabla}\cdot\vec{\omega})}}_{=0\ \circledast} - \underbrace{\cancel{\vec{\omega}(\overline{\nabla}\cdot\vec{v})}}_{=0\ \circledast}$$

$$-\vec{v}\cdot\overline{\nabla}\vec{\omega} + \vec{\omega}\cdot\overline{\nabla}\vec{v}$$

$\circledast$ These two terms are zero for incompressible flow since $\overline{\nabla}\cdot\vec{v} = \mathrm{DIV}(\vec{v}) = 0$ and $\overline{\nabla}\cdot\vec{\omega} = \overline{\nabla}\cdot(\overline{\nabla}\times\vec{v}) =$

$= \overline{\nabla}\times(\nabla\cdot\vec{v}) = 0$

Eq. [$\star$] can be finally recast as:

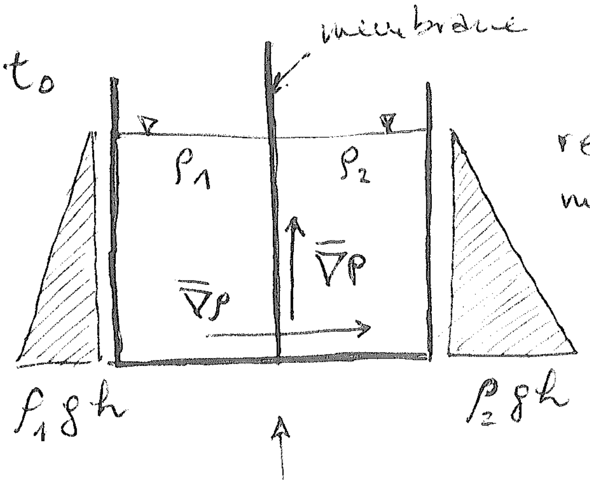
$$\boxed{\frac{\partial \vec{\omega}}{\partial t} + \underbrace{\vec{v} \cdot \bar{\nabla} \vec{\omega}}_{\substack{\text{CONVECTIVE} \\ \text{TRANSPORT} \\ \text{OF VORTICITY}}} = \frac{\bar{\nabla} \rho \times \bar{\nabla} P}{\rho^2} + \nu \bar{\nabla}^2 \vec{\omega} + \underbrace{\vec{\omega} \cdot \bar{\nabla} \vec{v}}_{\substack{\text{STRETCHING/} \\ \text{TILTING OF} \\ \text{VORTICITY}}}}$$

LAGRANGIAN DERIVATIVE OF VORTICITY : $\frac{D\vec{\omega}}{Dt}$

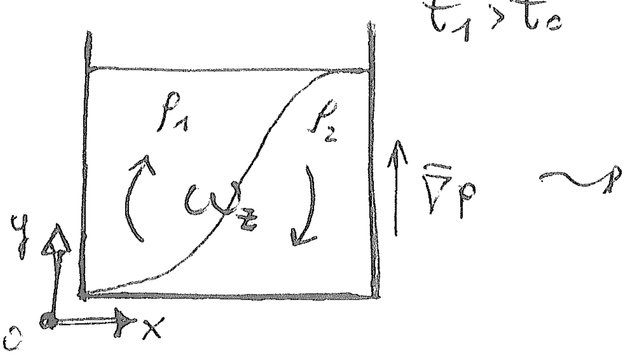
VORTICITY TRANSPORT EQUATION FOR A 3D INCOMPRESSIBLE FLUID

- The baroclinic term is different from zero only when a density gradient exists ($\bar{\nabla}\rho \neq 0$) and is orthogonal to the pressure gradient ($\bar{\nabla}\rho \perp \bar{\nabla}P$ so that $\bar{\nabla}\rho \times \bar{\nabla}P \neq 0$!). These conditions are typically met in the atmosphere when tornadoes and cyclones form : air temperature and pressure are such that density undergoes local changes and density gradient are thus formed; when these gradients are orthogonal to pressure gradients vorticity can be generated.

Working principle (qualitative explanation of baroclinicity): consider a tank filled with two liquids that have different densities (ρ_1 and $\rho_2 > \rho_1$) and are initially separated by an impermeable membrane :

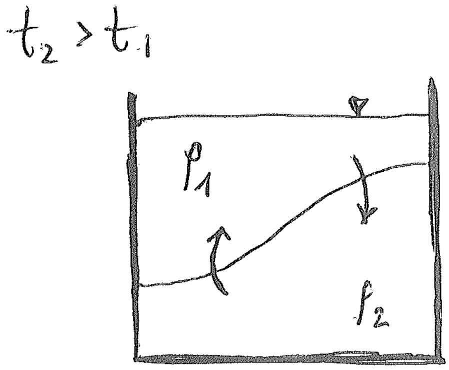


remove the membrane $\rightsquigarrow$

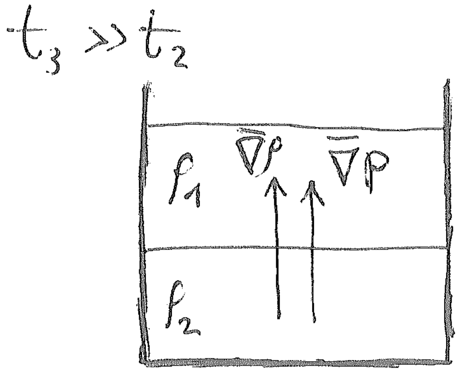


Once we remove the membrane a density gradient acts at the interface of the two liquids : $\bar{\nabla}\rho \perp \bar{\nabla} p$

The denser fluid tends to occupy the bottom of the tank and pushes the lighter fluid upwards : vorticity is generated



$\rightsquigarrow$



Final configuration

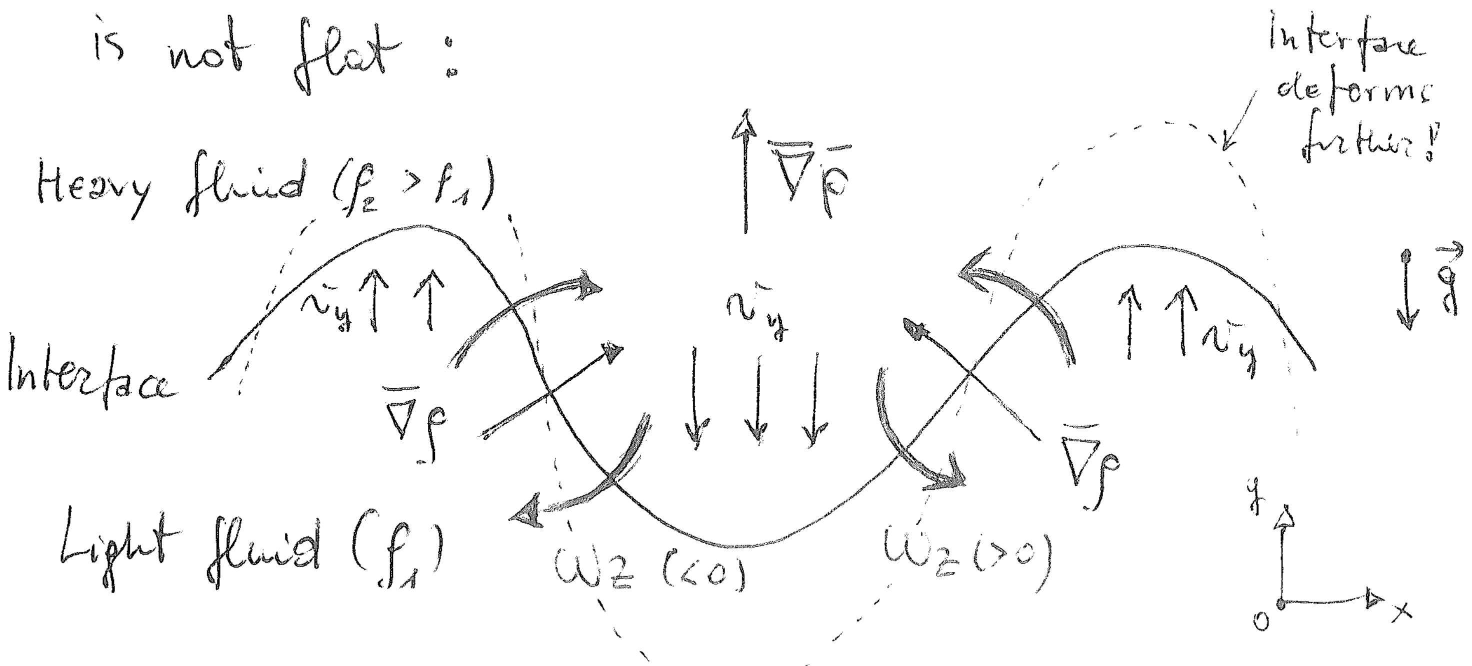
$\bar{\nabla}\rho \parallel \bar{\nabla} p$

After some oscillations, the motion of the two fluid stops and they form two static layers of lighter fluid on top of denser fluid : because $\bar{\nabla}\rho \parallel \bar{\nabla} p$ in this situation, then the baroclinic term has vanished and no vorticity is generated anymore.

This kind of instability of the interface occurs whenever density gradients and pressure gradients are misaligned at the perturbed interface.

In the previous example, misalignement decreases as the two fluid approach the final configuration: this means that less and less vorticity is generated (maximum vorticity generation occurs at the beginning) and the system is bound to reach a stable configuration.

In other situations, however, this may not be the case. One simple example is the following, where the interface between the two fluids is not flat:



Baroclinic vorticity is generated across the interface. This leads to the formation of two counter-rotating vortices that drag heavy fluid down-ward (in between the vortices) and light fluid up-ward (at the side of each vortex). This leads to additional generation of vorticity and further mis-alignment of the gradients (which tend to become "more perpendicular" to each other).

Note that, in the region where light fluid is driven upward by baroclinic vorticity, this light fluid is pushing the heavier fluid giving rise to the so-called RAYLEIGH-TAYLOR INSTABILITY of the interface.

Such instability is very common in environmental processes, such as ① viscous fingering with dissolution of CO_2 into brine in saline reservoirs or ② underwater oil spill in the ocean (when a layer of light oil forms below a layer of heavy salty water, subject to oceanic turbulence).

NOTE: How does viscous fingering work? When CO_2 is injected into underground saline reservoirs (for storage purposes), it partitions into a gas phase (gaseous CO_2) and a liquid phase (CO_2 dissolved in aqueous brine).

The gaseous CO_2 is lighter than brine and therefore can rise through the reservoir due to buoyancy; when the gaseous CO_2 encounters a low-permeability obstruction, it cannot rise further but rather starts moving laterally beneath the obstruction.

During this lateral motion, the gaseous CO_2 starts dissolving and becomes aqueous brine, which is up to 1% denser than CO_2-free brine. While gaseous CO_2 is stable below the obstruction and above the CO_2-free brine, the layer of CO_2-rich brine created by dissolution of gaseous CO_2 is not stable (above the CO_2-free brine) and a Rayleigh-Taylor instability sets in, leading to the formation of viscous fingers of "heavy" CO_2-rich brine alternated to fingers of "light" CO_2-free brine.

How to measure the "strenght" of the RT instability? It can be measured based on the density difference between the two fluids, through the so-called <u>ATWOOD NUMBER</u>:

$$\boxed{A = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}}$$

Based on this number, the <u>penetration distance</u> of the light fluid into the heavy fluid can be given as:

$$h = \alpha \cdot A \cdot g \cdot t^2$$

where α is a dimensionless parameter that quantifies the mixing rate of the process (namely how efficiently the two fluids are mixed by the instability).

- let us now go back to the transport equation at page 12 and focus on the <u>stretching/tilting term</u> $\vec{\omega} \cdot \nabla \vec{v}$.

In a 3D flow, this term can be written as:

x - component (of the vorticity transport eqn.) : $\vec{\omega} \cdot \vec{\nabla} \vec{v} \Big|_x = \underbrace{\omega_x \frac{\partial v_x}{\partial x}}_{\text{STRETCHING}} + \underbrace{\omega_y \frac{\partial v_x}{\partial y} + \omega_z \frac{\partial v_x}{\partial z}}_{\text{TILTING}}$

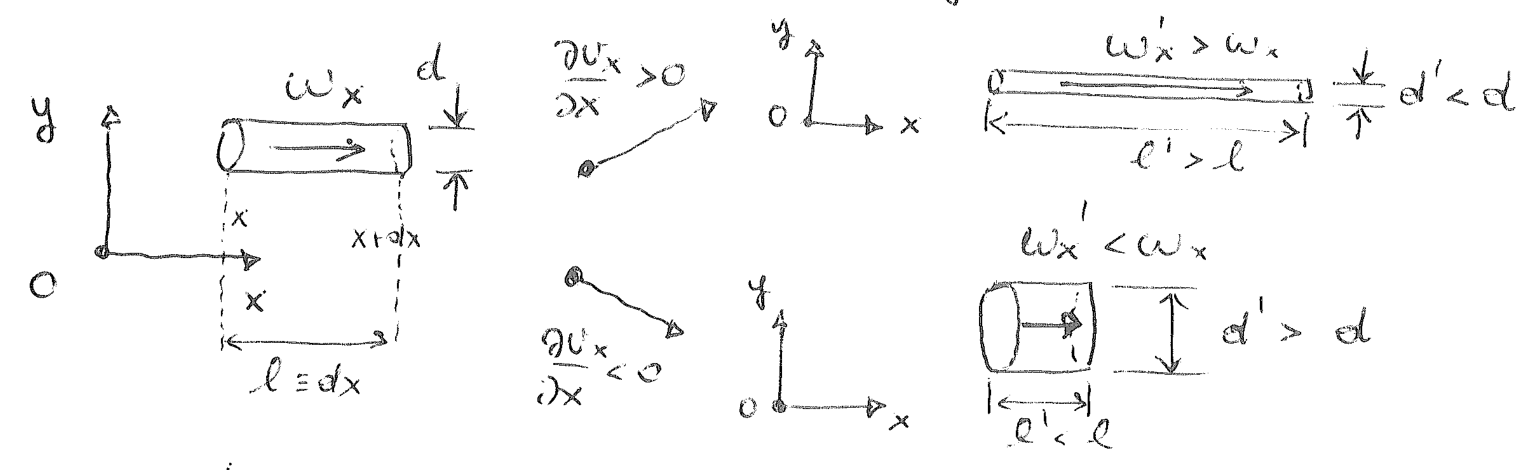
y - component : $\vec{\omega} \cdot \vec{\nabla} \vec{v} \Big|_y = \omega_x \frac{\partial v_y}{\partial x} + \omega_y \frac{\partial v_y}{\partial y} + \omega_z \frac{\partial v_y}{\partial z}$

z - component : $\vec{\omega} \cdot \vec{\nabla} \vec{v} \Big|_z = \omega_x \frac{\partial v_z}{\partial x} + \omega_y \frac{\partial v_z}{\partial y} + \omega_z \frac{\partial v_z}{\partial z}$

STRETCHING TILTING

Let us consider the x-component terms to explain stretching and tilting of vorticity. If $\omega_x \frac{\partial v_x}{\partial x} \neq 0$ then it must be $\omega_x \neq 0$ (let's assume $\omega_x > 0$) and $\frac{\partial v_x}{\partial x} \neq 0$.

So there will be a fluid element rotating at ω_x subject to a velocity gradient $\partial v_x / \partial x$:



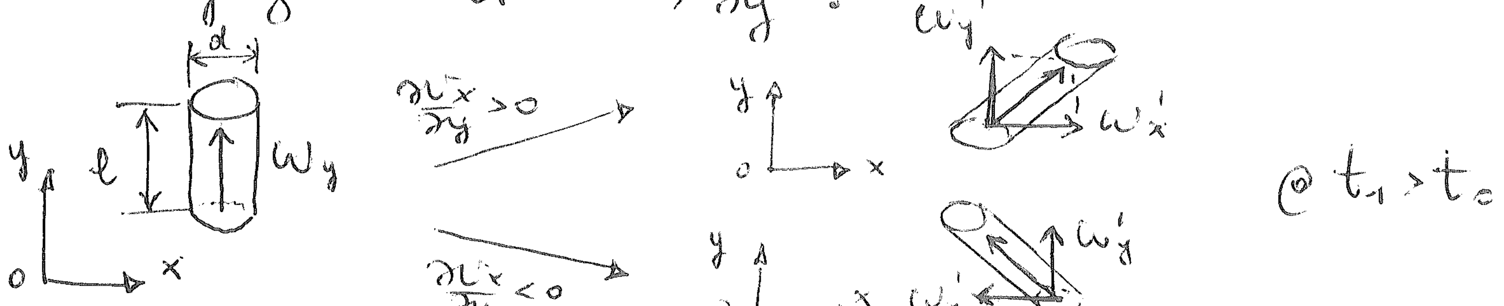
@ t_0 @ $t_1 > t_0$

- If $\frac{\partial v_x}{\partial x} > 0$ then $v_x(x+dx) > v_x(x)$ and the two ends of the fluid element will tend to move apart from each other as time goes on: the fluid element will thus be stretched (its final length l' will be larger than its initial length l). This has two consequences:
  (i) Vorticity will increase ($\omega_x' > \omega_x$) due to conservation of angular momentum associated to the fluid element → Vorticity is generated
  (ii) the cross-section of the element will decrease ($d' < d$) → New length scales are generated within the flow

- If $\frac{\partial v_x}{\partial x} < 0$ then $v_x(x+dx) < v_x(x)$ and the two ends of the fluid element will approach each other: the fluid element is now compressed (or negatively stretched) and its length decreases ($l' < l$). As a consequence vorticity decreases ($\omega_x' < \omega_x$, "loss" of vorticity) and the cross-section increases ($d' > d$, again new scales appear in the flow).

<u>Bottomline</u>: through the stretching term, vorticity can be created or destroyed within the flow.

Let us now consider the tilting term $\omega_y \frac{\partial v_x}{\partial y}$ (similar considerations would apply to the other tilting term $\omega_z \, \partial v_x / \partial z$). If $\omega_y \frac{\partial v_x}{\partial y} \neq 0$ then it must be $\omega_y \neq 0$ (assume $\omega_y > 0$) and $\partial v_x / \partial y \neq 0$, namely there will be a fluid element rotating at ω_y subject to a velocity gradient $\partial v_x / \partial y$:



- If $\frac{\partial v_x}{\partial y} > 0$ then $v_x(y+dy) > v_x$ and the upper end of the element travels in the x-direction faster than the lower end: as a result, the fluid element tilts and starts rotating not only along y but also along x. Part of the vorticity initially owned by the element (in y-direction) is partly converted into x-vorticity. In other words the total vorticity of the element is conserved, but the y-component is reduced ($\omega_y' < \omega_y$) and the difference is converted into ω_x vorticity.
- If $\frac{\partial v_x}{\partial y} < 0$ then $v_x(y+dy) < v_x$ and the element tilts in the opposite direction. It is still true, however, that part of the initial ω_y vorticity is converted into ω_x vorticity.

<u>Bottomline</u>: through the tilting terms, vorticity can be transferred from one (any) direction in space into another direction in space.

NOTE: $\vec{\omega}\cdot\bar{\nabla}\vec{v} \neq 0$ only in 3D flows

In 2D flow: $\vec{\omega}\cdot\bar{\nabla}\vec{v} = 0$ always

Example: suppose to have a flow in the x-y plane. Then the only vorticity component different from zero is ω_z. If $v_x \neq 0$ and $v_y \neq 0$ (but $v_z = 0$ in 2D) then:

$$\vec{\omega}\cdot\bar{\nabla}\vec{v}\Big|_x = \underbrace{\underset{\overset{\parallel}{0}}{\omega_x}\frac{\partial v_x}{\partial x}}_{=0} + \underbrace{\underset{\overset{\parallel}{0}}{\omega_y}\frac{\partial v_x}{\partial y}}_{=0} + \underbrace{\omega_z\underset{\overset{\parallel}{0}}{\frac{\partial v_x}{\partial z}}}_{=0}$$

$$\vec{\omega}\cdot\bar{\nabla}\vec{v}\Big|_y = \underbrace{\underset{\overset{\parallel}{0}}{\omega_x}\frac{\partial v_y}{\partial x}}_{=0} + \underbrace{\underset{\overset{\parallel}{0}}{\omega_y}\frac{\partial v_y}{\partial y}}_{=0} + \underbrace{\omega_z\underset{\overset{\parallel}{0}}{\frac{\partial v_y}{\partial z}}}_{=0}$$

etc...

So, there is no stretching nor tilting of vorticity in 2D flows, which are thus significantly different from 3D flows.

The vorticity transport equation in 2D for the case of constant density (no baroclinicity) reads as:

$$\boxed{\underbrace{\frac{\partial\vec{\omega}}{\partial t} + \vec{v}\cdot\bar{\nabla}\vec{\omega}}_{\frac{D\vec{\omega}}{Dt}} = \nu\,\bar{\nabla}^2\vec{\omega}}$$

VORTICITY TRANSPORT EQUATION FOR 2D INCOMPRESSIBLE FLOW