SIMILARITY OF CONSTITUTIVE TRANSPORT EQUATIONS

There is a similarity of the governing equations describing chemical, momentum and heat transport. Such similarity makes solutions of equations transferrable from one medium to the other.

We know that conservation of momentum for a Newtonian incompressible fluid yields the NS equations:

$$[1] \quad \rho\left(\frac{\partial \vec{u}}{\partial t} + \underbrace{\vec{u}\cdot\bar{\nabla}\vec{u}}_{\text{CONVECTIVE TERMS}}\right) = -\bar{\nabla}P + \underbrace{\mu\bar{\nabla}^2\vec{u}}_{\text{DIFFUSIVE TERMS}} \qquad \text{DIMENSIONAL}$$

$$\Downarrow$$

$$[2] \quad \frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\bar{\nabla}\vec{u} = -\bar{\nabla}P + \frac{1}{Re}\bar{\nabla}^2\vec{u} \qquad \text{NON-DIMENSIONAL}$$

Similarly, the general equation describing the mass transport of a given chemical is given by:

$$[3] \quad \frac{\partial c}{\partial t} + \underbrace{\vec{u}\cdot\bar{\nabla}c}_{\text{CONVECTIVE TERMS}} = \underbrace{D\bar{\nabla}^2 c}_{\text{DIFFUSIVE TERMS}} + R_c \qquad \text{DIMENSIONAL}$$

where c is the concentration of the chemi-cal and D is the diffusion coefficient. The term R_c is a reaction rate constant (which accounts for generation (source) or destruction (sink) of the chemical mass due to chemical reaction):

Typically, $R_c > 0$ means that the chemical reaction creates more of the chemical whereas $R_c < 0$ means that the chemical reaction destroys part of the chemical.

Note: R_c is a volumetric reaction rate.

If we assume, for simplicity, that $R_c = 0$ then the dimensionless form of eq [3] is:

$$[4] \quad \frac{\partial c}{\partial t} + \vec{u} \cdot \bar{\nabla} c = \frac{1}{Re \cdot Sc} \bar{\nabla}^2 c \qquad \text{NON-DIMENSIONAL}$$

where $\boxed{S_c \triangleq \nu / D = \frac{\text{MOMENTUM DIFFUSIVITY}}{\text{CHEMICAL DIFFUSIVITY}} \equiv \text{SCHMIDT NUMBER}}$

For the transport of heat, the following constitutive equation holds:

$$[5] \quad \rho c_p \Big(\frac{\partial \theta}{\partial t} + \underbrace{\vec{u} \cdot \bar{\nabla} \theta}_{\text{CONVECTION}} \Big) = \underbrace{K \bar{\nabla}^2 \theta}_{\text{DIFFUSION}} + \frac{Q}{\rho c_p} (\theta - \theta_{ref})$$

where c_p is the specific heat of the fluid, θ is the temperature of the fluid, K is the thermal conductivity of the fluid and Q is a heat source when $Q>0$ and a heat sink when $Q<0$.

Assuming $Q=0$ for simplicity, the dimensionless form of equation [5] is:

$$[6] \quad \frac{\partial \theta}{\partial t} + \vec{u} \cdot \bar{\nabla} \theta = \frac{1}{Re \cdot Pr} \bar{\nabla}^2 \theta$$

where $\boxed{Pr \triangleq \frac{\nu}{\alpha} = \frac{\text{MOMENTUM DIFFUSIVITY}}{\text{THERMAL DIFFUSIVITY}} \equiv \text{PRANDTL NUMBER}}$

and $\alpha = K/\rho C_p \equiv$ THERMAL DIFFUSIVITY $[m^2/s]$

NOTE: $[K] = \left[\frac{W}{mK}\right]$; $[C_p] = \left[\frac{J}{Kg \cdot K}\right]$

TRANSPORTED QUANTITY	DIFFUSION COEFFICIENT	VALUES *	
		AIR	WATER
MOMENTUM	ν	0.2	0.01
HEAT	α	0.2	$1.5 \cdot 10^{-3}$
CHEMICAL (CONCENTRATION)	D	0.2	10^{-5}

* Values at 20°C in cm^2/s.

The table tells us that, at ambient temperature, mass (concentration of chemical), momentum and heat are all transferred at approximately the same rate since all diffusion coefficients have very similar values. This is not true for water: Diffusion of momentum occurs at relatively higher rate than heat, and at much higher rate than diffusion of chemical concentration (read mass).

Similarity among constitutive transport equations is clear also when the 1-dimensional form of such equations is considered:

- 1D NS @ steady state (transport of momentum in the y direction due to fluid motion in the x direction):

$$\boxed{\tau_{xy} = \nu \frac{d(\rho v_x)}{dy}}$$

- 1D Fick law (diffusive mass transport of chemical along the y direction):

$$\boxed{J = - D \frac{dc}{dy}}$$

where J is the mass flux of chemical C $\left[\frac{kg}{m^2 s}\right]$

- 1D Fourier law (heat conduction along the y direction):

$$\boxed{q = -K \frac{\partial T}{\partial y}}$$

The first to recognize the similarity between mass, momentum and heat transport processes was Osborne Reynolds in 1874. In fact, Reynolds postulated that these processes are identical and therefore the following REYNOLDS ANALOGY can be written:

$$\underbrace{Re \cdot \frac{f}{2}}_{\text{MOMENTUM}} = \underbrace{Nu}_{\text{HEAT}} = \underbrace{Sh}_{\text{MASS}}$$

where f is the friction factor, and:

$$\boxed{Nu = \frac{h \cdot L}{K}}$$

NUSSELT NUMBER $\left(= \frac{\text{CONVECTIVE HEAT TRANS}}{\text{CONDUCTIVE HEAT TRANS.}}\right)$

with h = convective heat transfer coefficient, and

$$\boxed{Sh = \frac{K \cdot L}{D}}$$

SHERWOOD NUMBER $\left(= \frac{\text{CONVECTIVE MASS TRANS}}{\text{DIFFUSIVE MASS TRANS}}\right)$

with K = convective mass transfer coefficient [m/s].