STRATIFIED FLOWS

Almost all water bodies (oceans, lakes) are stratified, meaning that they are characterized by <u>density variations</u> (usually in the vertical direction).

Density variations exist most commonly because of temperature and/or salinity gradients. Stratification due to temperature gradients is the main contributor in fresh-water lakes. Stratification due to salinity gradients is the main contributor in oceans and in some inland lakes (e.g. the Great Salt Lake in Utah or the Dead Sea in Israel, where temperature gradients may also be present).

Stratification due to temperature gradients is the most interesting to model from a water quality point of view, because most biological and chemical reactions depend on temperature, as well as crucial processes like gas transfer across the air/water interface.

A parameter that is closely related to density (and density variations, in particular) is <u>BUOYANCY</u>:

$$\boxed{b \triangleq \frac{\rho_0 - \rho}{\rho_0} \cdot g}$$

BUOYANCY (AKA REDUCED GRAVITY)

where ρ is the density of a fluid parcel moving in the stratified flow, ρ_0 is the reference density of the fluid parcel (e.g. at ambient temperature) and g is gravitational acceleration.

Buoyancy, also known as reduced gravity, is defined so that the fluid parcel tends to rise when $b > 0$ (namely when $\rho < \rho_0$) and tends to settle when $b < 0$ (namely $\rho > \rho_0$).

If the density difference $\Delta\rho \triangleq \rho_0 - \rho$ occurs over a characteristic length scale h (usually vertical) in a variable-density flow characterized by a typical velocity U, the buoyancy can be written in non-dimensional form to obtain the RICHARDSON NUMBER:

$$\boxed{Ri = b \cdot \frac{h}{U^2}} = \boxed{\frac{\Delta\rho}{\rho} \cdot \frac{gh}{U^2}}$$

RICHARDSON NUMBER

Physical meaning:

$$Ri = \frac{\text{buoyancy forces}}{\text{inertial forces}}$$

From the above definition, it follows that buoyancy effects can occur either if $\Delta\rho/\rho$ is significant (as, for instance, in small-scale plumes) or if $gh/\rho U^2$ is significant (as, for instance, in large-scale geophysical flows).

Flows created entirely by buoyancy forces are referred to as NATURAL-CONVECTION FLOWS, and are distinct from FORCED-CONVECTION FLOWS, which are driven primarily by pressure gradients.

Clearly, many flows are subject to a combination of natural and forced convection.

One example that is relevant to environmental problems is provided by industrial smokestacks, which typically have a significant imposed momentum flux to assist the initial rise of the contaminant plume.

Another example is provided by the effluent from chemical and power plants, which usually enters a river with sufficient momentum to carry it away from the bank and toward the center of the water stream before buoyancy forces can carry the contaminant to the surface or bottom of the river.

Buoyancy can be treated like any other state variable, such as temperature itself or concentration. Therefore, buoyancy obeys a conservation equation of the form:

$$[1] \quad \frac{\partial b}{\partial t} + \underbrace{\vec{v} \cdot \bar{\nabla} b}_{\text{CONVECTIVE TERM}} = \underbrace{\bar{\nabla} \left(K_b \bar{\nabla} b \right)}_{\text{DIFFUSIVE TERM}} + \text{Source/Sink}$$

where K_b is diffusivity for buoyancy and the source/sink term accounts for possible causes of buoyancy such as:

1) Solar heating at air/water interfaces, which generates buoyancy in a temperature-stratified system
2) evaporation, which may cause (negative) buoyancy near the surface of a water body due either to cooling (heat loss due to evaporation) or to increased salinity in the case of saline water.

In addition to the conservation equation [1], an equation of state can be formulated, considering that:

$$\boxed{\rho = \rho(T, C, p)}$$

where T = temperature, C = concentration of species dissolved in the fluid subject to buoyancy and p = pressure.

Typically C is concentration of salt in water, namely salinity S so that one can put $C = S$.

Another assumption that can be made, which is valid in many situations except in the deep oceans or in the atmosphere, is incompressibility, namely density is not dependent on pressure. Based on this assumption:

$$\rho = \rho(T, S)$$

The dependence of ρ on T and S can be expressed through the following coefficients:

[2] $$\boxed{\alpha = -\frac{1}{\rho_0} \cdot \frac{\partial \rho}{\partial T}}$$ THERMAL EXPANSION COEFFICIENT (AKA VOLUMETRIC EXP COEFF.)

[3] $$\boxed{\beta = \frac{1}{\rho_0} \cdot \frac{\partial \rho}{\partial S}}$$ SALINE EXPANSION COEFFICIENT

In general, both α and β are functions of T and S, respectively. However, approximate values that can be assumed are:

$$\alpha \cong 2 \cdot 10^{-4} \ {}^\circ C^{-1}$$

$$\beta \cong 7{,}5 \cdot 10^{-3} \ W\% \text{ (weight percent)}$$

These relations agree with the fact that salinity has a much greater effect on density than temperature.

Using eqns [2] and [3], density can be written as:

$$[4] \quad \boxed{\rho = \rho_0 (1 - \alpha \cdot \Delta T + \beta \cdot \Delta S)}$$

EQUATION OF STATE FOR DENSITY

where $\Delta T = T - T_0$ and $\Delta S = S - S_0$. Here:

T_0 = reference value for temperature (usually $T_0 = 4°C$)

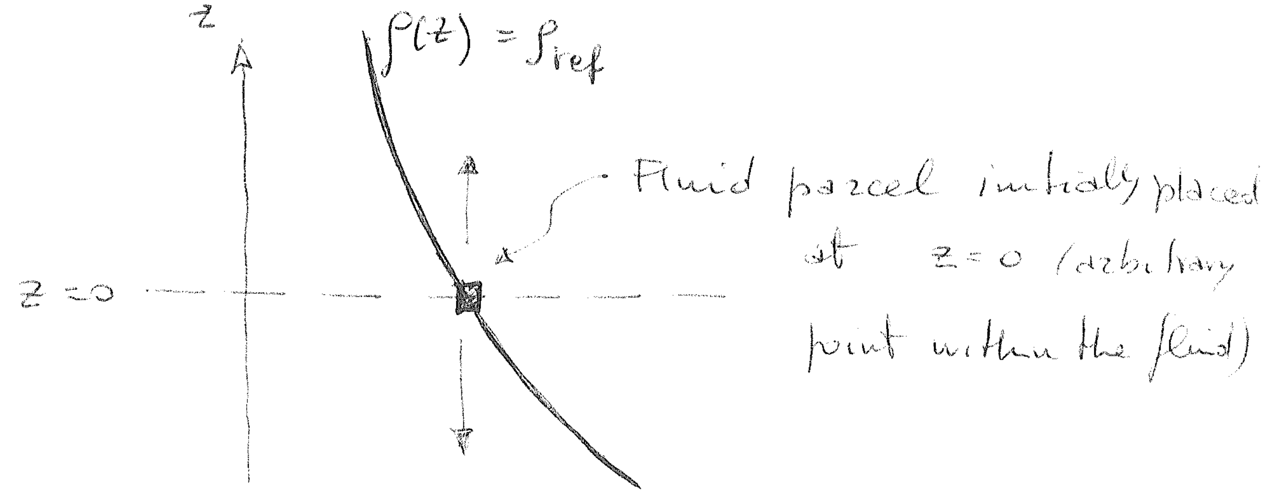
S_0 = reference value for salinity (usually $S_0 = 0\%$)

Note that $\rho = \rho_0$ when $T = T_0$ and $S = S_0$!

GRAVITATIONAL STABILITY

When the flow of a fluid is characterized by density differences, an important concept is gravitational stability.

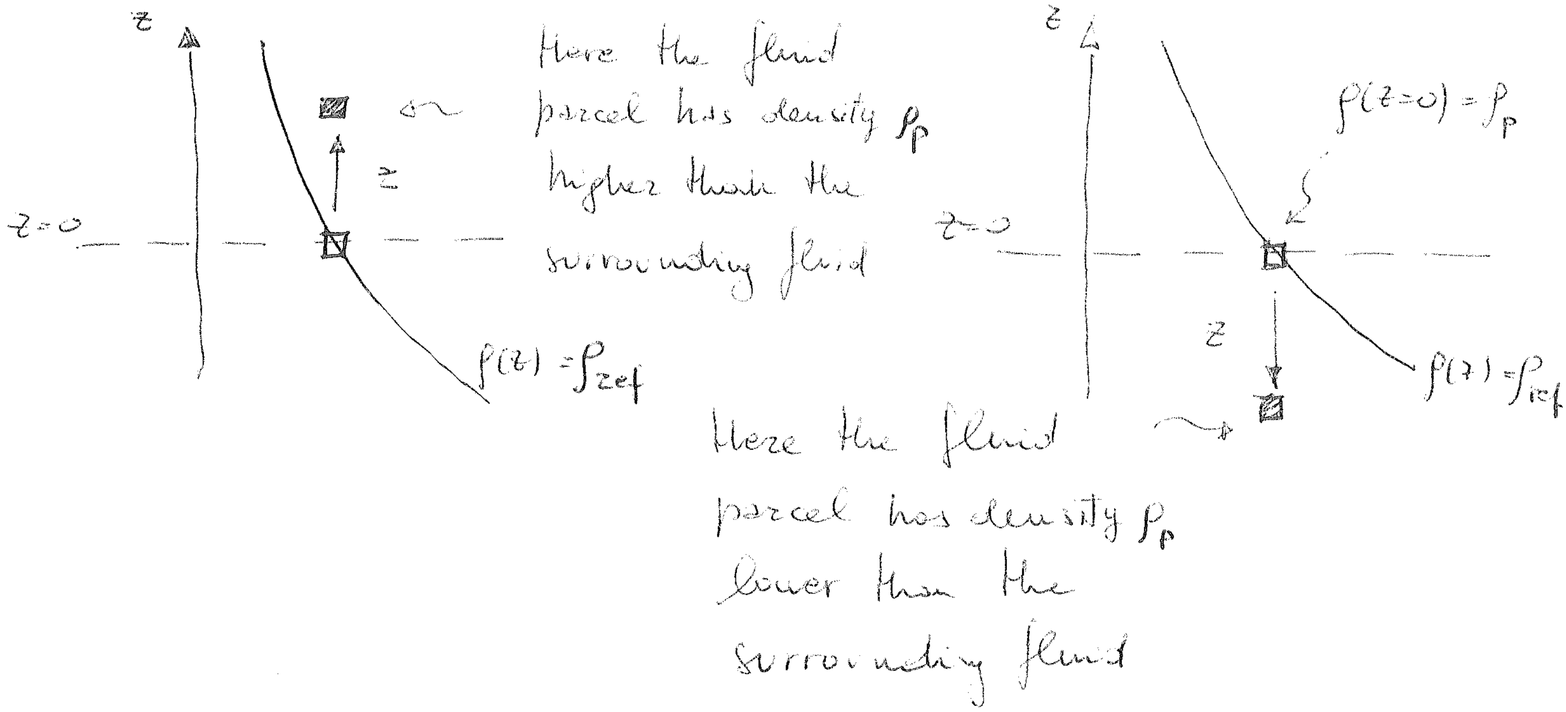
Gravitational stability means that the fluid system is stable only when heavier fluid underlies lighter fluid. If this does not happen (like we've already seen in the case of Rayleigh-Taylor instability), then buoyancy generates instabilities in the fluid system (associated with the tendency of the lighter fluid to rise). In turn, these instabilities give rise to convective motions that act to mix the fluid vertically. This can be demonstrated mathematically for a situation in which the fluid system is initially at rest (all fluid velocity components are equal to zero) and only density stratification is present:



For a fluid at rest, statics tells us that:

$$\frac{\partial p}{\partial z} = -\rho(z) \cdot g$$

If the fluid parcel is displaced vertically from its initial position by a <u>small amount</u> z', then a buoyancy force acting on the fluid parcel arises due to the density difference between the parcel and its new surroundings:



The buoyancy force acting on the parcel, per unit volume is $-g(\rho_p - \rho_{ref})$ where:

ρ_p = density of the fluid parcel

ρ_{ref} = density of the surrounding fluid

Recalling Newton's law:

$$\vec{F} = m\vec{a} \longrightarrow \underbrace{\frac{\vec{F}}{V}}_{-g(\rho_p - \rho_{ref})} = \underbrace{\frac{m}{V}}_{\rho_p} \underbrace{\vec{a}}_{\frac{d^2 z}{dt^2}}$$

where V = volume of the fluid parcel, one gets:

$$[5] \quad -g(\rho_p - \rho_{ref}) = \rho_p \frac{d^2 z}{dt^2}$$

If we now express ρ_p and ρ_{ref} through a Taylor series expansion, then we get:

$$\rho_p = \rho_0 + z \frac{d\rho_p}{dz} + \underbrace{E[O(z^2)]}_{\text{Truncated terms (2nd order)}}$$

[6]

$$\rho_{ref} = \rho_0 + z \frac{d\rho_{ref}}{dz} + E[O(z^2)]$$

Substituting the expressions given in eq. [6] into eq. [5], one gets:

$$-g\left(\cancel{\rho_0} + z\frac{d\rho_p}{dz} - \cancel{\rho_0} - z\frac{d\rho_{ref}}{dz}\right) = \rho_p \frac{d^2 z}{dt^2}$$

$$g \cdot z\left(\frac{d\rho_{ref}}{dz} - \frac{d\rho_p}{dz}\right) = \rho_p \frac{d^2 z}{dt^2}$$

Considering now that the relationship between pressure and density in the absence of heat transfer can be expressed as:

$$dp = c^2 d\rho$$

where c = sound speed, one also gets:

$$\frac{\partial p}{\partial z} = -\rho g \;\Rightarrow\; c^2 \frac{d\rho}{dz} = -\rho g \;\Rightarrow\; \boxed{\frac{d\rho}{dz} = -\frac{\rho g}{c^2}}$$

$dp = c^2 d\rho$

Assuming that $\rho_p \cong \rho_0$, one thus finds:

$$\frac{d\rho_p}{dz} \cong \frac{d\rho_0}{dz} = -\frac{\rho_0 g}{c^2}$$

$$\Downarrow$$

$$g \cdot z \left(\frac{d\rho_{ref}}{dz} + \frac{\rho_0 g}{c^2} \right) = \rho_p \frac{d^2 z}{dt^2}$$

$\rho_p \simeq \rho_0$!

The following quantity can be now introduced:

[7] $$\boxed{N = \left(-\frac{g}{\rho_0} \cdot \frac{d\rho_{ref}}{dz} \right)^{1/2}}$$

BRUNT-VAISALA (or BUOYANCY) FREQUENCY (definition valid in water flows)

$$\Downarrow$$

$$N^2 = -\frac{g}{\rho_0} \cdot \frac{d\rho_{ref}}{dz}$$

where it is assumed that $d\rho_{ref}/dz < 0$!

Upon substitution of eq. [7] into the equation for d^2z/dt^2, one gets:

$$\underbrace{g \cdot z\left(-\frac{\rho_0 N^2}{g} + \rho_0 \frac{g}{c^2}\right)}_{-\rho_0 \cdot z \cdot N^2 + \rho_0 \cdot z \frac{g^2}{c^2}} = \rho_p \frac{d^2 z}{dt^2}$$

negligible since c is usually very large!

$$-\cancel{\rho_0} \cdot z N^2 = \cancel{\rho_p} \frac{d^2 z}{dt^2}$$

$$\rho_0 \simeq \rho_p$$

$$\boxed{\frac{d^2 z}{dt^2} + N^2 \cdot z = 0}$$

This equation is the equation of a simple harmonic motion, and has solutions of the form:

STABLE CONDITION $\quad z \propto e^{iNt} \quad$ for $N^2 > 0$

UNSTABLE CONDITION $\quad z \propto e^{|N|t} \quad$ for $N^2 < 0$

NEUTRAL CONDITION $\quad z \propto e^{0} \quad$ for $N^2 = 0$

The case $\boxed{N^2 > 0}$ represents a gravitationally stable condition in which density increases with depth, so lighter fluid is on top of heavier fluid ($d\rho/dz < 0$).

The case $\boxed{N^2 < 0}$ represents a gravitationally unstable condition in which density decreases with depth, so that heavier fluid is on top of lighter fluid ($d\rho/dz > 0$ and N is undefined). Such unstable condition leads to convection.

The case $\boxed{N^2 = 0}$ represents a gravitationally neutral condition in which $d\rho/dz = 0$ so there is no buoyancy force acting on the fluid parcel and, therefore, there is no acceleration* d^2z/dt^2 when the fluid parcel is displaced from its initial position.

* vertical acceleration!

Elaborating on the $N^2 \neq 0$ cases:

$\boxed{N^2 > 0} \Rightarrow \quad z \propto e^{iNt} = \cos(Nt) + i\,\mathrm{sen}(Nt)$

STABLE STRATIFICATION

The fluid parcel, perturbed vertically from its starting position, experiences a vertical acceleration back towards the initial position. Because of such acceleration, the parcel oscillates vertically around the initial position. This oscillatory motion occurs at a frequency given by N, and oscillations are seen to decay over time (due to viscous effects). Eventually, oscillations fade away and the parcel returns to its initial position.

$\boxed{N^2 < 0} \Rightarrow$ The fluid parcel, perturbed vertically from its starting position, experiences a vertical acceleration away from the initial position. Because of such acceleration, the particle oscillates

UNSTABLE STRATIFICATION

vertically away from the starting position, and oscillations grow exponentially with time. Given the initial small perturbation z, particle position grows exponentially leading to the onset of large-scale convective motions (called INTERNAL GRAVITY WAVES in stratified water flows)

NOTE: The Brunt-Väisälä frequency can be also defined in gas flows (e.g. in the atmosphere):

$$N \triangleq \left(\frac{g}{\theta} \cdot \frac{d\theta_{ref}}{dz} \right)^{1/2}$$

where θ is the POTENTIAL TEMPERATURE

$$\theta \triangleq T \left(\frac{p_0}{p} \right)^{R/c_p}$$

with T the actual temperature [K], R the gas constant for air and c_p the specific heat capacity at a constant pressure.

The potential temperature of a fluid parcel in air is the temperature that such parcel would have if brought to a standard reference pressure P_0 (usually 1 bar) from its actual pressure p.

For air : $R/c_p \cong 0.286$ (POISSON CONSTANT)

Why is potential temperature used? Because it represents a line of thermodynamic equilibrium in which there is no net exchange of energy with the surrounding environment (e.g. through radiative absorption/emission or through turbulent mixing). So if an air parcel moves (rising, sinking or translating horizontally) without exchanging energy with the surrounding air parcels, then θ is constant. All parcels lying on a constant-θ surface are therefore in local thermodynamic equilibrium.