II) TURBULENT B.L. ON A FLAT PLATE

In the region where the B.L. is turbulent the following procedure can be followed. Starting from the NS_x in the case $\partial p/\partial x = 0$, one can integrate such equation along the vertical direction y, from the wall $(y=0)$ to a distance far away from it $(y \to \infty)$:

$$\underbrace{\int_0^\infty v_x \frac{\partial v_x}{\partial x} dy}_{A} + \underbrace{\int_0^\infty v_y \frac{\partial v_x}{\partial y} dy}_{B} = \nu \underbrace{\int_0^\infty \frac{\partial^2 v_x}{\partial y^2} dy}_{C} \qquad [11]$$

Integral B can be recast as:

$$\int_0^\infty v_y \frac{\partial v_x}{\partial y} dy = \underbrace{v_x \cdot v_y \Big|_0^\infty}_{\underbrace{v_x(y\to\infty)}_{= v_\infty} \cdot v_y(y\to\infty) - \cancelto{0}{v_x(y=0) \cdot v_y(y=0)}} - \int_0^\infty v_x \frac{\partial v_y}{\partial y} dy$$

$$= v_\infty \cdot v_y(y \to \infty) - \int_0^\infty v_x \frac{\partial v_y}{\partial y} dy$$

Now, what is $v_y(y \to \infty) = ?$ We can find out using continuity:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \Rightarrow \frac{\partial v_y}{\partial y} = -\frac{\partial v_x}{\partial x} \Rightarrow d v_y = -\frac{\partial v_x}{\partial x} dy$$

$$\underbrace{\int_{v_y(y=0)}^{v_y(y\to\infty)} d v_y}_{v_y(y\to\infty) - \cancel{v_y(y=0)} = 0} = -\int_0^\infty \frac{\partial v_x}{\partial x} dy \Rightarrow \boxed{v_y(y \to \infty) = -\int_0^\infty \frac{\partial v_x}{\partial x} dy}$$

The condition $\frac{\partial v_y}{\partial y} = -\frac{\partial v_x}{\partial x}$ also yields:

$$-\int_0^\infty v_x \frac{\partial v_y}{\partial y} dy = +\int_0^\infty v_x \frac{\partial v_x}{\partial x} dy$$

Integral B can thus be rewritten as:

$$\int_0^\infty v_y \frac{\partial v_x}{\partial y} dy = -v_\infty \int_0^\infty \frac{\partial v_x}{\partial x} dy + \int_0^\infty v_x \frac{\partial v_x}{\partial x} dy$$

Integral C can be recast as:

$$\mu \int_0^\infty \frac{\partial^2 v_x}{\partial y^2} dy = \mu \cdot \frac{\partial v_x}{\partial y}\Big|_0^\infty = \mu \left(\underbrace{\cancel{\frac{\partial v_x}{\partial y}\Big|_{y\to\infty}}}_{=0\ (v_x = v_\infty = \text{CONST} @ y\to\infty)} - \frac{\partial v_x}{\partial y}\Big|_{y=0} \right) =$$

$$= -\nu \left.\frac{\partial v_x}{\partial y}\right|_{y=0} = -\frac{\mu}{\rho} \left.\frac{\partial v_x}{\partial y}\right|_{y=0} = -\frac{1}{\rho}\cdot \tau_w$$

with τ_w = shear stress (viscous component) at the wall ($y=0$). Eq. (11) becomes:

$$-v_\infty \int_0^\infty \frac{\partial v_x}{\partial x}\,dy + \int_0^\infty v_x \frac{\partial v_x}{\partial x}\,dy + \int_0^\infty v_x \frac{\partial v_x}{\partial x}\,dy = -\frac{\tau_w}{\rho}$$

$$\boxed{\frac{\tau_w}{\rho}} = \underbrace{v_\infty \int_0^\infty \frac{\partial v_x}{\partial x}\,dy}_{} - \underbrace{2\int_0^\infty v_x \frac{\partial v_x}{\partial x}\,dy}_{}$$

$$\int_0^\infty v_\infty \frac{\partial v_x}{\partial x}\,dy = \int_0^\infty \frac{\partial (v_x v_\infty)}{\partial x}\,dy \qquad \int_0^\infty 2 v_x \frac{\partial v_x}{\partial x}\,dy = \int_0^\infty \frac{\partial v_x^2}{\partial x}\,dy$$

$$= \int_0^\infty \frac{\partial (v_x v_\infty)}{\partial x}\,dy - \int_0^\infty \frac{\partial v_x^2}{\partial x}\,dy$$

$$= \int_0^\infty \left[\frac{\partial}{\partial x}\left(v_x v_\infty - v_x^2\right)\right] dy$$

INVERT $\int$ WITH $\frac{\partial}{\partial x}$

$$= \frac{\partial}{\partial x}\int_0^\infty \left(v_x v_\infty - v_x^2\right) dy = \boxed{\frac{\partial}{\partial x}\int_0^\infty \frac{v_x}{v_\infty}\left(1 - \frac{v_x}{v_\infty}\right) v_\infty^2\,dy}$$

$$\frac{\tau_w}{\rho} \underset{\uparrow}{=} \frac{\partial}{\partial x}\left[\underbrace{\int_0^\infty \frac{v_x}{v_\infty}\left(1 - \frac{v_x}{v_\infty}\right) dy}\right] \cdot v_\infty^2$$

$v_\infty = \text{CONST}!$

This quantity is defined as MOMENTUM THICKNESS

$$\boxed{\hat{\delta} \triangleq \int_0^\infty \frac{v_x}{v_\infty}\left(1 - \frac{v_x}{v_\infty}\right) dy}$$ MOMENTUM THICKNESS

$$\Downarrow$$

$$\tau_w = \rho v_\infty^2 \frac{\partial \hat{\delta}}{\partial x}$$

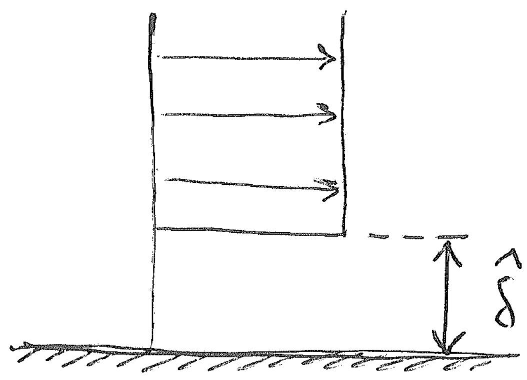
$\hat{\delta} = \hat{\delta}(x) \rightsquigarrow$

$$= \rho v_\infty^2 \frac{d\hat{\delta}}{dx} \qquad [12]$$

Physical meaning of the momentum thickness: if there was no B.L. (namely no region of the flow where viscosity matters), then the flow would be potential everywhere and there would be no "need" to satisfy the no-slip condition: the fluid velocity could be equal to v_∞ every-where. In reality, this does not happen and the fluid velocity is smaller than v_∞ inside the B.L. This implies that, compared to an ideal case of negligible viscosity effects, there will be a loss (or, better, reduction) of momentum:

$$\frac{\dot{m}}{W} \triangleq \int_0^{\infty} \rho v_x (v_\infty - v_x)\, dy \quad \circledast$$

One can imagine a fictitious (not physical) situation in which the velocity profile has the following shape:



$$v_x = \begin{cases} v_\infty & \text{for } y \geq \hat{\delta} \\ 0 & \text{for } y < \hat{\delta} \end{cases}$$

$\hat{\delta}$ is the distance by which the velocity profile should be displaced along y (and away from the plate) to generate the same reduction of momentum as in the real B.L.:

$$\frac{\dot{m}}{W} \triangleq \rho v_\infty^2 \cdot \hat{\delta} \quad \oplus$$

Equalling $\circledast$ and $\Diamond$ one gets:

$$\cancel{\rho}\, v_\infty^2\, \hat{\delta} = \int_0^\infty \cancel{\rho}\, v_x (v_\infty - v_x)\, dy$$

$$\hat{\delta} = \int_0^\infty \frac{v_x}{v_\infty}\left(1 - \frac{v_x}{v_\infty}\right) dy \qquad \underbrace{\text{Q.E.D.}}_{\text{Quod Erat Demonstrandum}}$$

Now: $\eta = y/\delta(x) \rightarrow y = \eta \cdot \delta(x) \rightarrow \boxed{dy = \delta(x)\, d\eta}$

Hence:

$$\hat{\delta} = \int_0^\infty \frac{v_x}{v_\infty}\left(1 - \frac{v_x}{v_\infty}\right) dy =$$

$$= \int_0^{\delta(x)} \frac{v_x}{v_\infty}\left(1 - \frac{v_x}{v_\infty}\right) dy + \underbrace{\int_{\delta(x)}^\infty \frac{v_x}{v_\infty}\left(1 - \frac{v_x}{v_\infty}\right) dy}_{=0 \text{ because } v_x = v_\infty \text{ for } y > \delta(x)!}$$

If $y = \delta(x)$ then $\eta = 1$ $\longrightarrow$

$$= \int_0^1 \frac{v_x}{v_\infty}\left(1 - \frac{v_x}{v_\infty}\right) \delta(x)\, d\eta$$

and eqn. [12] at page 20 becomes:

$$\tau_w = \rho v_\infty^2 \frac{d}{dx}\left[\delta(x) \underbrace{\int_0^1 \frac{v_x}{v_\infty}\left(1 - \frac{v_x}{v_\infty}\right) d\eta}_{\text{THIS TERM IS INDEPENDENT OF X}}\right]$$

In conclusion:

$$\tau_w = \rho v_\infty^2 \frac{d\delta(x)}{dx} \int_0^1 \frac{v_x}{v_\infty}\left(1 - \frac{v_x}{v_\infty}\right) d\eta$$

This equation can be used to determine the B.L. growth in the turbulent region.

From τ_w one can define the <u>SHEAR VELOCITY</u>:

$$v_\tau \triangleq \sqrt{\frac{\tau_w}{\rho}} \qquad (\tau_w = \rho u_\tau^2)$$

This velocity has no straightforward physical meaning: Close to the wall, τ_w is important and it is used to define a velocity scale that is appropriate in the near-wall region.

Experimental observations and numerical simulations have shown that, in a turbulent B.L.:
(over a flat plate)

$$\boxed{\frac{v_x}{v_\tau} = 8.74 \left(\frac{y \cdot v_\tau}{\nu}\right)^{1/7}} \qquad [13]$$

Eq. [13] must be valid also for $y = \delta(x)$:

$$@\ y = \delta(x) \Rightarrow \frac{v_x}{v_\tau} \underset{\substack{\uparrow \\ v_x(y=\delta(x)) = v_\infty}}{=} \frac{v_\infty}{v_\tau} = 8{,}74 \left(\frac{\delta(x) \cdot v_\tau}{\nu} \right)^{1/7}$$

One thus gets:

$$\boxed{\frac{v_x}{v_\infty}} = \frac{v_x}{v_\tau} \cdot \frac{1}{\left(\frac{v_\infty}{v_\tau}\right)}$$

$$= \cancel{8{,}74} \left(\frac{y \cdot \cancel{v_\tau}}{\cancel{\nu}} \right)^{1/7} \cdot \frac{1}{\cancel{8{,}74} \left(\frac{\delta(x) \cdot \cancel{v_\tau}}{\cancel{\nu}} \right)^{1/7}}$$

$$= \left(\frac{y}{\delta(x)} \right)^{1/7} = \boxed{\eta^{1/7}} \quad [14]$$

Going back to the equation for τ_w, eq. [14] yields:

$$\tau_w = \rho v_\infty^2 \frac{d\delta(x)}{dx} \overbrace{\int_0^1 \eta^{1/7} (1 - \eta^{1/7})\, d\eta}^{= 7/72}$$

$$\tau_w = \frac{7}{72}\,\rho\, v_\infty^{\,2}\,\frac{d\delta(x)}{dx} \qquad [15]$$

Combining eq. [13] with $v_\tau = \sqrt{\frac{\tau_w}{\rho}}$ one also gets:

$$\frac{v_\infty}{v_\tau} = 8.74\left(\frac{\delta(x)\cdot v_\tau}{\nu}\right)^{1/7}$$

$$v_\infty = 8.74\left(\frac{\delta(x)}{\nu}\right)^{1/7}\cdot v_\tau^{\,8/7}$$

$v_\tau = \sqrt{\frac{\tau_w}{\rho}} \rightsquigarrow$

$$= 8.74\left(\frac{\delta(x)}{\nu}\right)^{1/7}\cdot\left(\frac{\tau_w}{\rho}\right)^{4/7}$$

$$\tau_w = \left[\frac{v_\infty}{8.74\left(\frac{\delta(x)}{\nu}\right)^{1/7}}\right]^{7/4}\cdot\rho$$

$$= \rho\,\underbrace{\left(\frac{1}{8.74}\right)^{7/4}}_{0{,}0225}\cdot\left[\frac{\nu}{\delta(x)}\right]^{1/4}\cdot v_\infty^{\,7/4}$$

$$= 0{,}0225\,\rho\left[\nu/\delta(x)\right]^{1/4}\cdot v_\infty^{\,7/4} \qquad [16]$$

Equalling eq. [15] and eq. [16]:

$$\frac{7}{72}\cancel{\rho} v_\infty^2 \frac{d\delta(x)}{dx} = 0{,}0225\cancel{\rho}\left[\frac{\nu}{\delta(x)}\right]^{1/4}\cdot v_\infty^{7/4}$$

$$\frac{d\delta(x)}{dx} \cong 0{,}23\left[\frac{\nu}{\delta(x)}\right]^{1/4}\cdot v_\infty^{1/4}$$

Upon seperation of variables:

$$\int_0^{\delta(x)} d\delta(x)\cdot\delta(x)^{1/4} \cong 0{,}23(\nu/v_\infty)^{1/4}\cdot\int_0^x dx$$

↑

We assume that the laminar region of the B.L. is short (this is true, actually!) so that the turbulent region starts at $x \simeq 0$, when $\delta(x)$ is still small and one can put $\delta \simeq 0$!

$$\frac{4}{5}\delta(x)^{5/4} \cong 0{,}23\ (\nu/v_\infty)^{1/4}\cdot x$$

$$\delta(x) = \underbrace{\left(\frac{5}{4}\cdot 0{,}23\right)^{4/5}}_{\simeq 0{,}37}\cdot(\nu/v_\infty)^{1/5} x^{4/5}$$

In the turbulent region of the B.L. $\delta(x) \propto x^{4/5}$

Since $\delta(x) = 0{,}37\,(\nu/v_\infty)^{1/5}\cdot X^{4/5}$ one also finds, from eq. [16]:

$$\tau_w = 0{,}0225\,\rho\left[\frac{\nu}{\delta(x)}\right]^{1/4}\cdot v_\infty^{7/4}$$

$$= \frac{0{,}0225\,\rho\,\nu^{1/4}\,v_\infty^{7/4}}{\left[0{,}37\,(\nu/v_\infty)^{1/5}\,X^{4/5}\right]^{1/4}}$$

$$\cong 0{,}0287\,\rho\,v_\infty^2\left(\frac{\nu}{v_\infty\cdot X}\right)^{1/5}$$

namely $\tau_w \propto X^{-1/5}$ in turbulent conditions.

The expressions developed in this section for $\delta(x)$ and τ_w are applicable for values of $Re_x \triangleq \frac{v_\infty\cdot X}{\nu}$ up to about 10^7. For higher values of the Reynolds number, more complicated velocity profiles should be considered.

EXAMPLE OF CALCULATION OF $\delta(x)$ AND τ_w:

$v_\infty = 1$ m/s

$\nu = 1{,}57\cdot 10^{-5}$ m²/s (AIR) $\Longrightarrow$

$X = 10$ m

$$\begin{cases} \delta(x) = 0{,}2555 \text{ m} \stackrel{!}{\cong} 25{,}5 \text{ cm} \\ \tau_w \cong 2{,}58\cdot 10^{-3}\ \dfrac{N}{m^2} \end{cases}$$

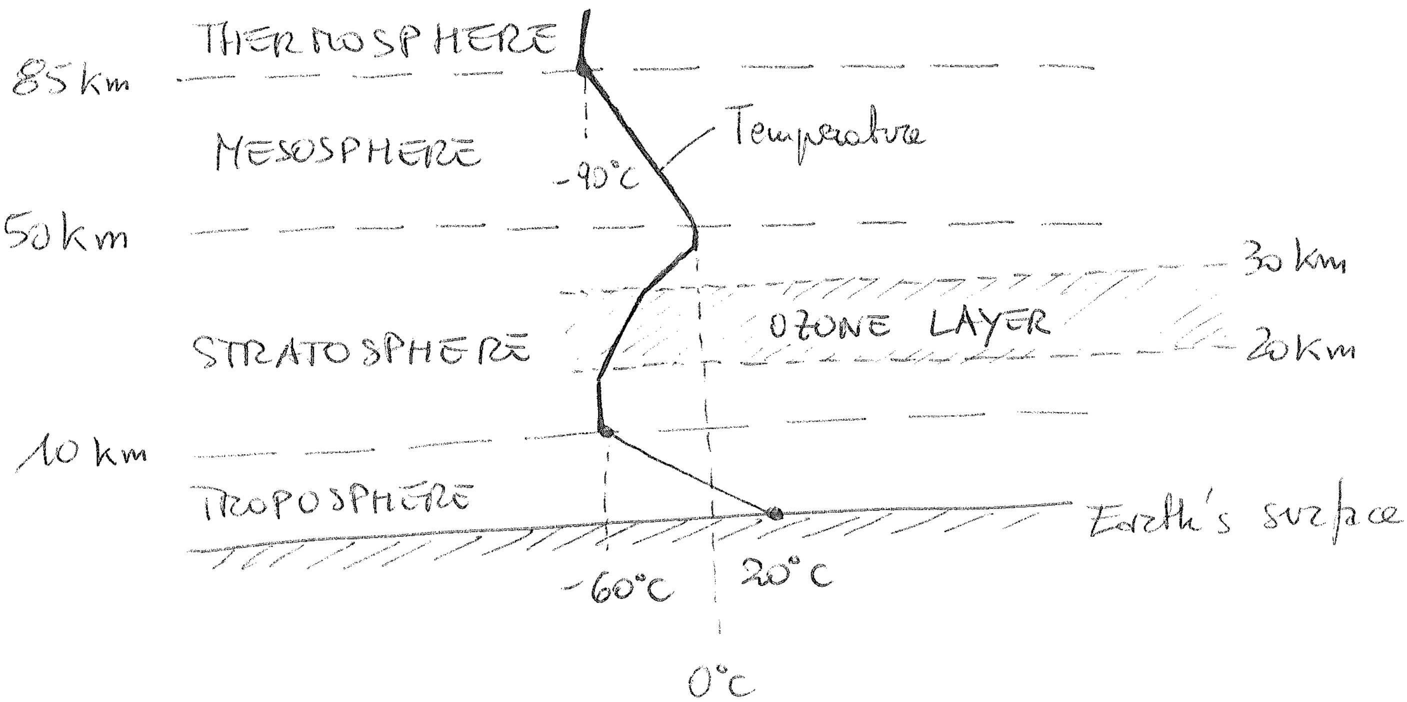
With water ($\nu = 10^{-6}\ m^2/s$) :

$$\delta(x = 10\,m) = 0{,}1473\,m \simeq 14{,}73\,cm$$

$$\tau_w(x = 10\,m) = 1{,}14\ N/m^2$$

ATMOSPHERIC BOUNDARY LAYER (aka PLANETARY BOUNDARY LAYER)

The Earth's atmosphere is more than 100 km thick and is typically divided into layers :



The lowest portion of the atmosphere, the troposphere, is vital to life on Earth. From an environmental point of view, the troposphere is important because this is where all man-made gaseous emissions are discharged. From a meteorological point of view, troposphere is important because all weather Systems (cyclones, anticyclones, storms and hurricanes) are concentrated in it.

Troposphere is a complex system, characterized by complex physical processes and instabilities related to turbulence: "Thermal" turbulence accounts for convection during the day and stratification at night, "mechanical" turbulence accounts for wind stirring (which displaces large masses of air), cloud formation and rain precipitation, disper= sion and dilution of pollutants.
Other complicating factors are provided by the orography of the surface (due to the presence of buildings, forests, hills, mountains) and the formation of large weather events, which make the troposphere permanently turmoiled.

Within the troposphere, closest to the ground, lies the Atmospheric Boundary Layer (ABL), which is about 1 km thick and represents the layer

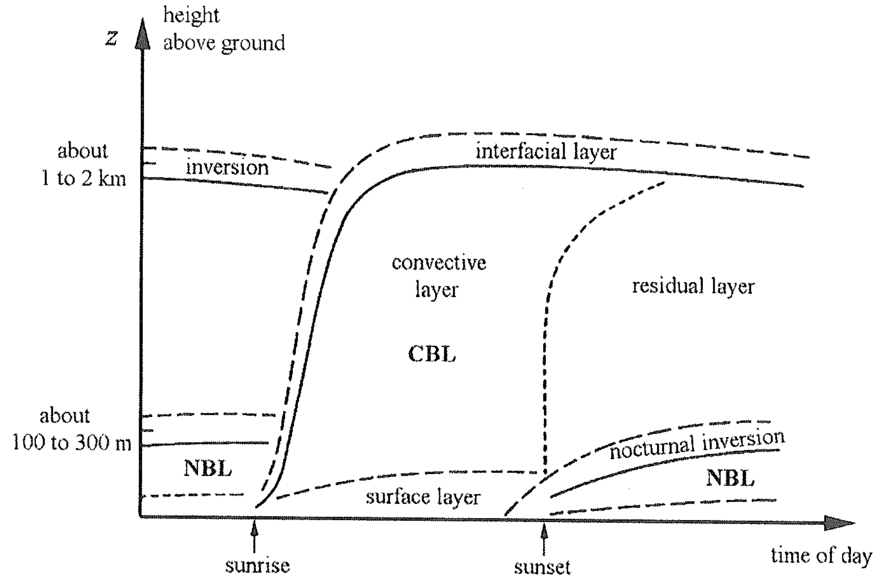
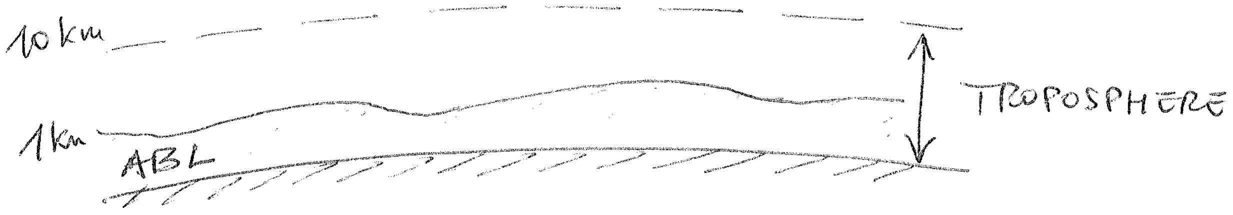


Figure 12.1: Typical evolution of the atmospheric boundary layer (ABL) over the course of a day over land and under clear skies. At sunrise, heating from below sets to a convective boundary layer (CBL), while at sunset heat loss to space terminates convection and creates a thin nocturnal boundary layer (NBL). [Adapted from Garratt, 1992]

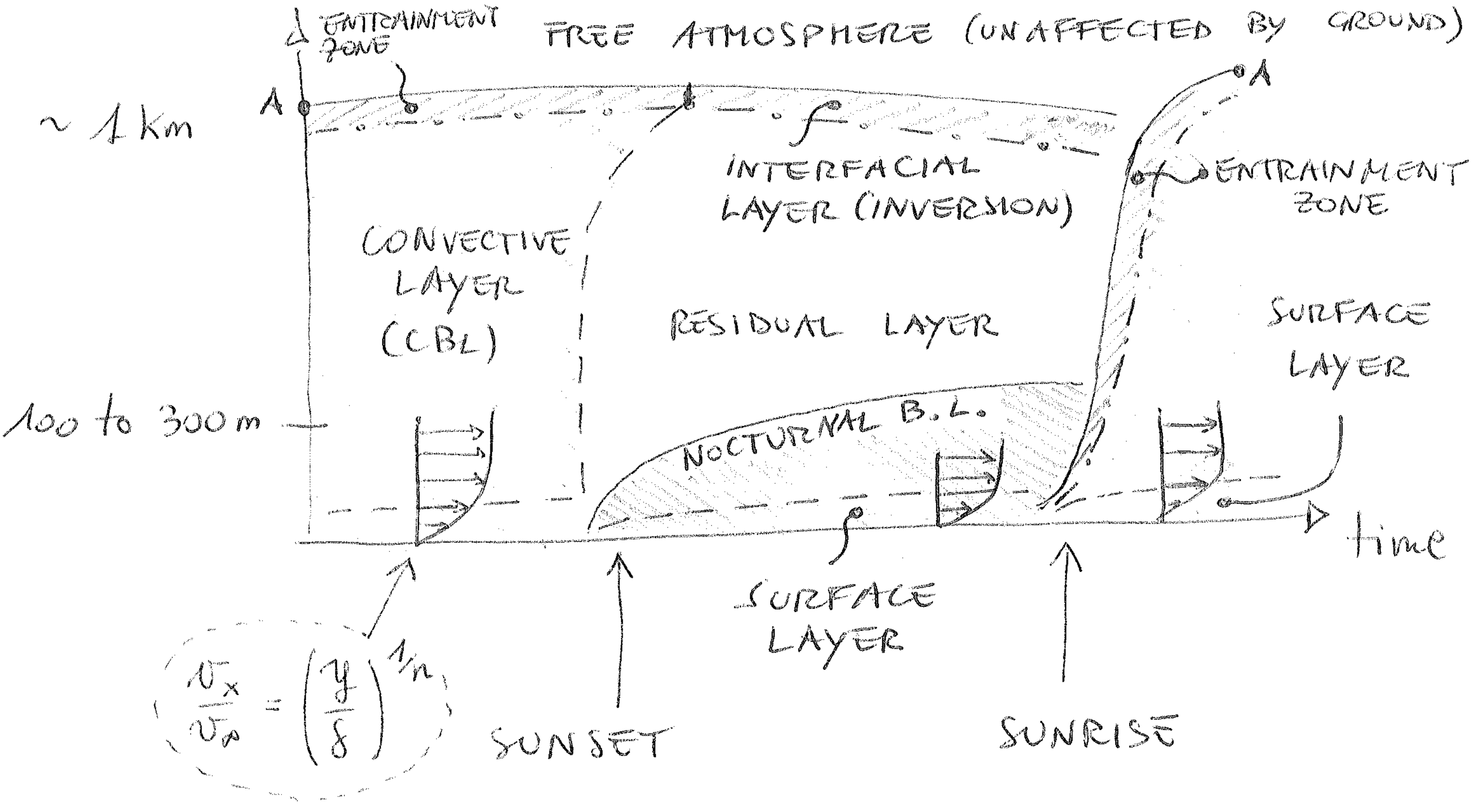
create weather patterns, which cause precipitation, water the crops and provide a freshwater supply to us on land. Second, turbulent mixing generates dispersion and dilution of our pollutants.

Within the troposphere, closest to the ground, lies the *Atmospheric Boundary Layer* (ABL). It is about 1 km thick and forms the layer where the atmosphere feels the contact with the ground surface, land or sea. The surface-air interaction occurs in two primary forms: mechanical and thermal. The mechanical contact arises from the friction exerted by the wind against the ground surface; this friction causes the wind to be sheared and creates turbulence. In the absence of thermal processes, i.e. when the ABL is said to be *neutral*, we expect a logarithmic velocity profile $u(z)$ (see Section 8.2) characterized by the friction velocity u_* and the roughness height z_o. The friction velocity is intimately related to the level of turbulence in the lowest atmosphere.

The thermal contact between the lower atmosphere and the ground surface has its origin in the solar radiation. Sun light is electromagnetic radiation in the visible range, to which the atmosphere is largely transparent. (We see through the air over fairly long distances.) Therefore, much of the solar radiation traverses the atmosphere and reaches the surface. Land surface, by contrast, is where the atmosphere feels the presence of the ground (land or sea surface):



The ABL evolves in time over the course of a day, giving rise to the following situation:



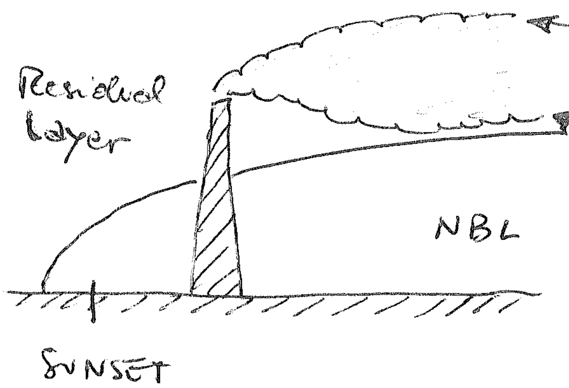
At sunrise, the atmosphere begins to be heated from below (land surface absorbs solar radiation and radiates heat back into the atmosphere): Convective motions set in and lead to the formation of a <u>CONVECTIVE BOUNDARY LAYER</u> (CBL). Inside the CBL, convection-induced turbulence dominates over wind-induced turbulence except in the lower region of the CBL, called surface layer: Inside the surface layer, wind-induced turbulence balances convection but wind can generate strong velocity gradients (namely strong shear).

When the sun sets, the heat released by the land surface decreases and convective motions gradually fade away: The CBL becomes a quiet, unagitated layer called residual layer,* below which a shallow <u>NOCTURNAL BOUNDARY LAYER (NBL)</u> forms. In the absence of convection, thermal stratification** in the vertical direction builds up during night-time: temperature increases with height and this damps mixing, therefore the NBL is quite stable.

* Pollutants can be trapped inside the residual layer for many hours.

** Nocturnal thermal stratification is gradually eroded during day-time by convection.

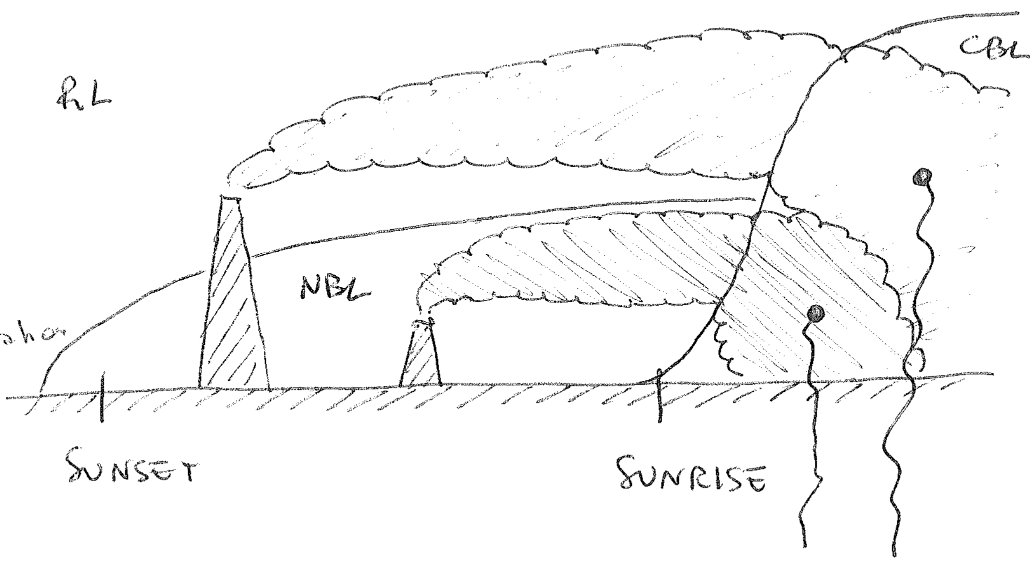
Effect of the stable NBL on smoke plumes:



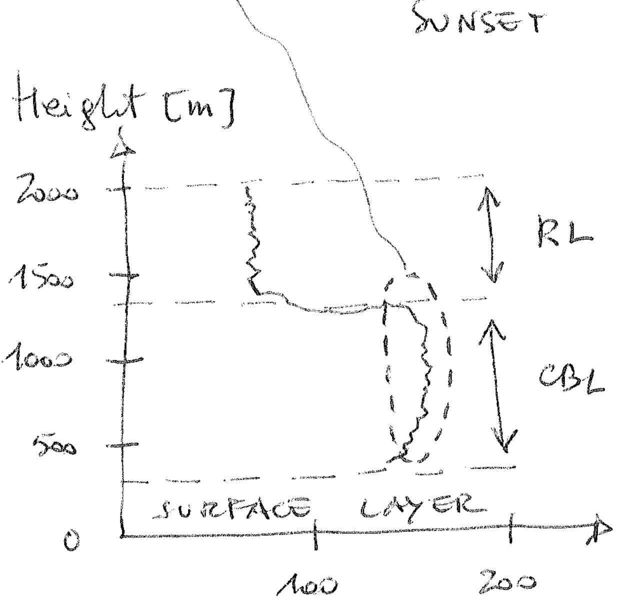
The top of the plume can grow upward inside the residual layer

The bottom of the plume is stopped by the stable nocturnal layer

Effect of the CBL on smoke plumes:



Mixing produces nearly uniform values of concentration across the CBL



Particle concentration across the CBL and across the residual layer

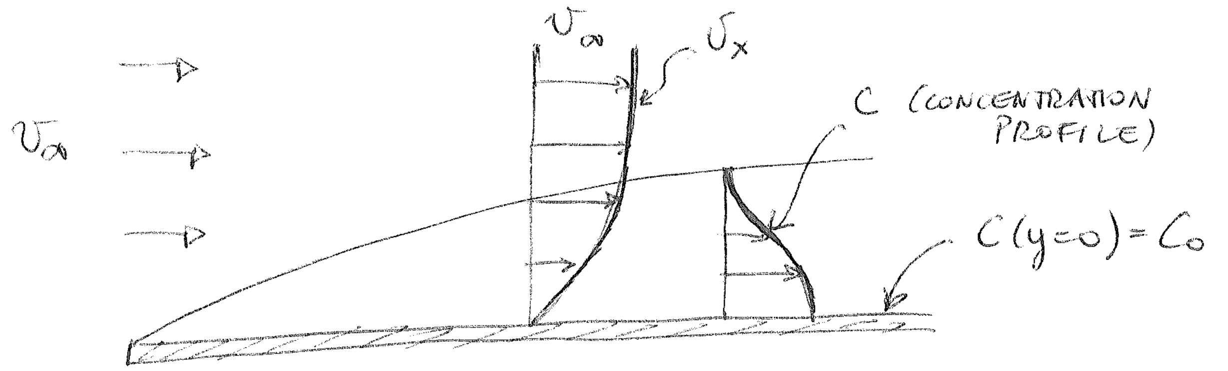
The CBL mixes the smoke plumes, entraining particulate matter down to the ground

Mixing inside the CBL generates high values of particulate concentration (and much lower values of concentration inside the upper residual layer). Therefore pollution tends to be well mixed across the CBL but remains almost completely confined within the CBL.

ONE FINAL NOTE ON THE BLASIUS SOLUTION FOR THE LAMINAR B.L. DEVELOPING ON A FLAT PLATE

One of the few cases in which it is possible to study analitically mass transport inside a B.L. is precisely the 2D steady laminar B.L. over a flat plate.

Consider for instance the case of mass transport of a chemical species C from a flat plate into a fluid inside the laminar region of a Blasius B.L.



The solution obtained for the wall shear stress:

$$\tau_w = 0,332\,\mu\sqrt{\frac{v_\infty^3}{\nu}}\cdot x^{-1/2}$$

$$\stackrel{!}{=} 0,332\,\mu\frac{v_\infty}{x}\sqrt{\frac{v_\infty\cdot x}{\nu}}$$

$$Re_x \triangleq \frac{v_\infty\cdot x}{\nu} \rightsquigarrow \quad \stackrel{!}{=} 0,332\,\mu\frac{v_\infty}{x}\,Re_x^{1/2}$$

allows to formulate a solution to the transport of mass with boundary conditions $C(y=0)=C_0$, namely the chemical species has a non-zero concentration at the wall that diffuses inside the B.L., and $C(y\to\infty)=0$.

The solution is only possible when the mass diffusion coefficient D is equal to the momentum diffusion coefficient ν, namely when their ratio (the well-known SCHMIDT NUMBER) is equal to unity:

$$Sc = \frac{\nu}{D}$$

Air is characterized by $Sc \simeq 1$ and therefore the solution is valid for air. Water is characterized by $Sc \gg 1$ and therefore the solution does not apply to water.

If $Sc = 1$ then one can obtain:

$$\boxed{J_c = -D_c \frac{dC}{dy}}$$

ONE-DIMENSIONAL FLUX OF CHEMICAL C
(NOTE: J IS A MASS FLUX)

MASS FLUX OF CHEMICAL C [$kg/m^2 s$]

MASS DIFFUSION COEFFICIENT FOR CHEMICAL C [m^2/s]

CONCENTRATION GRADIENT OF CHEMICAL C [$\frac{kg}{m^3}$]

with $\frac{dC}{dy}$ given by numerical integration of the following conservation equation for mass:

$$v_x \frac{\partial C}{\partial x} + v_y \frac{\partial C}{\partial y} = D \frac{\partial^2 C}{\partial y^2} \quad [17]$$

Indeed, using the similarity condition $v_x = v_\infty \cdot f'(\eta)$ with $\eta = y/\delta(x)$ and $\delta(x) = \sqrt{\frac{\nu \cdot x}{v_\infty}}$, one can rewrite equation [17] as:

$$C'' + \frac{1}{2} \cdot \frac{\nu}{D} f \cdot C' = 0$$

If $\nu/D = 1$ and $C = f'$, then the above equation for concentration reads as:

$\longrightarrow$

$$f''' + \frac{1}{2} f \cdot f'' = 0$$

and this equation coincides with the one already solved for velocity (yet with different boundary conditions: $f'(\eta = 0) = C_0$ and $f'(\eta \to \infty) = 0$).

This yields:

$$\left.\frac{\partial c}{\partial y}\right|_{y=0} = -\left(\frac{0,332\, Re_x^{1/2}}{x}\right) C_0$$

$$J_c\Big|_{y=0} = \left(\frac{0,332\, D_c\, Re_x^{1/2}}{x}\right) C_0$$